Monads and sets. On Gödel, Leibniz, and the reflection principle

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Abstract

Gödel once offered an argument for the general reflection principle in set theory that took the form of an analogy with Leibniz' Monadology. I discuss the mathematical and philosophical background to Gödel's argument, reconstruct the proposed analogy in detail, and argue that it has no justificatory force.

Voor Göran, in dank en vriendschap

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1 Introduction

Gödel described his general philosophical theory to Hao Wang as "a monadology with a central monad [...] like the monadology of Leibniz in its general structure".[1] At the same time, he believed that Cantorian set theory is a true theory, which describes some "well-determined reality".[2] I will first discuss the embedding of Cantorian set theory in a Leibnizian metaphysics that the combination of these two beliefs of Gödel's requires.[3] Then I turn to an attempt by Gödel to justify (a particular form of) the reflection principle in set theory by drawing an analogy to the monadology. Of this attempt I will argue that, although its success might not depend on whether the monadology is true or not, it fails. More generally, I defend the claim that while a Leibnizian metaphysics is compatible with Cantorian set theory, by itself it provides no clues that can be used in justifying set-theoretical principles, be it by analogy or directly.[4]

2 Fitting Cantor's sets into Leibniz' metaphysics

One immediate obstacle to the project of relating Cantorian set theory to Leibniz' metaphysics in any positive way would seem to be this. Cantor defines a set as a "many, which can be thought of as a one"[5] and as "each gathering-together ['Zusammenfassung'] M into a whole of determined and well-distinguished objects m of our intuition or of our thought (which are called the 'elements' of M)".[6] Cantorian set theory being largely about infinite sets, it is a theory of certain infinite wholes. But Leibniz denies the existence of infinite wholes of

[1] Wang [1996], 0.2.1.

[2] Gödel [1990], p.181.

[3] Paul Benacerraf kindly allowed me to relate the following. At a dinner in 1974 or 1975, Gödel had conversations with Gerald Sacks on large cardinals and with Benacerraf on the mind-body problem. In the latter, he made reference to "monads". Gödel carried on these two conversations *simultaneously*, turning from left to right and back. (One argument advanced by Gödel was this: (1) the monads that our minds are have unambiguous access to the full set-theoretic hierarchy; (2) the full set-theoretic hierarchy cannot be adequately represented physically; therefore, (3) the mind cannot be reduced to a physical structure.)

[4] A monograph on the monadology in relation to Cantorian set theory is Osterheld-Koepke [1984]. However, the reflection principle is not discussed there. On another note, it is argued there (p.128) that on monadological grounds we can never decide the Continuum Hypothesis; one may well doubt that Gödel's understanding of the monadology and its relation to set theory would have had such a consequence. Gödel paired his belief in the monadology to a conviction that in principle a rational mind could decide every mathematical proposition. (He believed that "Leibniz did not in his writings about the Characteristica universalis speak of a utopian project" and that this would provide a means "to solve mathematical problems systematically", Gödel [1990], p.140. He realized that, because of his own incompleteness theorem, such a Characteristica could not assume the form of an entirely formal system.) In particular, he worked hard (but unsuccessfully) at deciding the Continuum Hypothesis. For further discussion of Gödel's belief in the solvability of all mathematical problems, see Kennedy & Van Atten [2004].

[5] Cantor [1932] p.204n.1.

[6] Cantor [1932], p.282; trl. modified from Grattan-Guinness [2000], p.112.

any kind.[7] For example, he says that one has to acknowledge that there are infinitely many numbers,[8] but he denies that they can be thought of as forming a unity:

> "I concede [the existence of] an infinite multitude, but this multitude forms neither a number nor one whole. It only means that there are more elements than can be designated by a number, just as there is a multitude or complex of all numbers; but this multitude is neither a number nor one whole".[9]

The distinction Leibniz draws between aggregates that are unities and aggregates that are mere multitudes is somewhat similar to the one Cantor would later draw between sets and proper classes, but their reasons are very different. Leibniz arrives at this distinction by a general argument that would rule out any infinite set altogether. He argues that there can be no infinite wholes or unities of any kind. It is not the notion of infinity as such that poses the problem for him, as is clear from this exchange between Philalèthe and Théophile (who represents Leibniz) in the *New Essays*:

> "PH: We have no idea of an infinite space, and nothing is clearer than the absurdity of an actual idea of an infinite number.
>
> TH: I agree. But the reason for this is not that one could have no idea of the infinite, but that an infinity cannot be a true whole."[10]

Specifically, Leibniz holds that the notion of an infinite whole contradicts the axiom that the whole is greater than the part. It is well known that Leibniz' argument is not sound and rests on an equivocation on "greater than", once defined in terms of the notion of proper superset and once defined in terms of the notion of non-surjective injection.[11] It can be shown, although for limitations of space I will not do so here, that Leibniz himself had all the means to see that his argument is not sound. The importance of that fact is that it shows

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[7] Friedman [1975], p.338, suggests that even so, Leibniz might have been willing to accept the for him inconsistent notion of infinite whole as a fiction that may prove useful in calculations, on a par with his acceptance of imaginary roots in algebra. To illustrate this point, Friedman refers to Leibniz [1705], II, ch.17, §3.

[8] Leibniz to Des Bosses, March 11/17, 1706, Leibniz [1875–1890], II, p.305: "One cannot deny that the natures of all possible numbers are indeed given, at least in God's mind, and that as a consequence the multitude of numbers is infinite." ("Neque enim negari potest, omnium numerorum possibilium naturas revera dari, saltem in divina mente, adeoque numerorum multitudinem esse infinitam.") Where translations are my own, I give the original as well.

[9] Leibniz to Joh. Bernoulli, February 21, 1699, Leibniz [1849–1863], III/2, p.575: "Concedo multitudinem infinitam, sed haec multitudo non facit numerum seu unum totum; nec aliud significat, quam plures esse terminos, quam numero designari possint, prorsus quemadmodum datur multitudino seu complexus omnium numerorum; sed haec multitudo non est numerus, nec unum totum."

[10] Leibniz [1875–1890], V, p.146: "PH : Nous n'avons pas l'idée d'un espace infini, et rien n'est plus sensible que l'absurdité d'une idée actuelle d'un nombre infini. TH : Je suis du même avis. Mais ce n'est pas parcequ'on ne sauroit avoir l'idée de l'infini, mais parcequ'un infini ne sauroit estre un vrai tout."

[11] See, for example, the refutation in Benardete [1964], pp.47–48.

that Leibniz' denial of infinite wholes does not reflect a limitation intrinsic to his philosophical system.

In Gödel's notebooks, I have so far not found a specific comment on Leibniz' argument that there can be no infinite wholes. But in the Russell paper from 1944 he wrote:

> "Nor is it self-contradictory that a proper part should be identical (not merely equal) to the whole, as is seen in the case of structures in the abstract sense. The structure of the series of integers, e.g., contains itself as a proper part".[12]

Among other things, Gödel says here that it is consistent that an equality relation holds between proper part and the whole. This entails a rejection of Leibniz' argument. And in a very similar note from 1944, again without mentioning Leibniz, Gödel adds: "the same can be contained as a part in 2 different ways".[13] That same consideration can be used to show that Leibniz' argument is not valid. Of course, the incorrectness of Leibniz' argument against infinite wholes implies nothing as to whether its conclusion is true or false. But clearly it will not be this argument that poses an obstacle to combining, as Gödel did, a belief in monadology with a belief in Cantorian set theory.

I now turn to the status of pure sets in Leibniz' metaphysics itself. Leibniz calls collections "aggregates" or "multitudes". In his philosophical remarks on them, he usually discusses aggregates of objects in the world; but from these remarks together with what he says about pure numbers, one can derive what his philosophical views on pure sets would have been.[14]

In a letter to De Volder of 1704, Leibniz writes that

> "Whatever aggregates out of pluralities there are, they are *unities* only in thought. They have no other reality than a borrowed one or that of the things out of which they are composed".[15]

Note the similarity with Cantor's definitions of a set that were quoted in the previous section; there with an emphasis on sets being a "one" or a "whole", here on the fact that for Leibniz as for Cantor, the unity of an aggregate consists its elements being thought or considered together. Therefore, Leibniz says, an aggregate has the character of a relation:

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[12] Gödel [1990], p.130.

[13] "dasselbe [kann] auf 2 verschiedene Weisen als Teil enthalten sein", Gödel's Notebook *Max XI* (1944), p.18.

[14] Gödel makes some remarks on monads and sets on Wang [1996], p.296, but not so much on the relation between them.

[15] January 21, 1704, Leibniz [1875–1890], II, p.261: "quaecunque ex pluribus aggregata sunt, ea non sunt unum nisi mente, nec habent realitatem aliam quam mutuatam seu rerum ex quibus aggregantur." Also Leibniz [1705], p.133: "Cette unité de l'idée des Aggregés est tres veritable, mais dans le fonds il faut avouer que cette unité des collections n'est qu'un rapport ou une relation dont le fondement est dans ce qui se trouve en chacune des substances singulieres à part. Ainsi ces Estres par Aggregation n'ont point d'autre unité achevée que la mentale ; et par consequent leur Entité aussi est en quelque façon mentale ou de phenomene, comme celle de l'arc en ciel."

> "Being and one are reciprocal notions, but where a being is given by aggregation, we also have one being, even though that entity and that unity are semi-mental.
>
> Numbers, units, and fractions have the nature of relations. And to that extent, they may in a sense be called beings."[16]

Leibniz here qualifies a unified aggregate as a semi-mental entity because he is thinking of aggregates of objects in the world. But an aggregate of mental objects would be entirely mental. The pure sets as we know them from Cantor's set theory, then, for Leibniz would fundamentally be pure relations that are entirely in the mind. Not in the human mind, but in God's mind, for, as Leibniz writes in the *New Essays*:

> "The relations have a reality that is dependent on the mind, as do truths; but not on the human mind, as there is a supreme intelligence that determines all of them at all times".[17]

Correspondingly, the truths about these pure relations have their existence in God's mind:

> "One must not say, with some Scotists, that the eternal verities would exist even though there were no understanding, not even that of God.
>
> For it is, in my judgement, the divine understanding which gives reality to the eternal verities, albeit God's will have no part therein. All reality must be founded on something existent. It is true that an atheist may be a geometrician: but if there were no God, geometry would have no object. And without God, not only would there be nothing existent, but there would be nothing possible. That, however, does not hinder those who do not see the connexion of all things with one another and with God from being able to understand certain sciences, without knowing their first source, which is in God".[18]

And, in "On the radical origination of things" from 1697:

> "Neither these essences nor the so-called eternal truths about them are fictitious but exist in a certain region of ideas, if I may so call it, namely, in God himself, who is the source of all essence and of the existence of the rest [...] and since, furthermore, existing things

[16] Leibniz to Des Bosses, March 11, 1706, Leibniz [1875–1890], II, p.304: "Ens et unum convertuntur, sed ut datur Ens per aggregationem, ita et unum, etsi haec Entitas Unitasque sit semimentalis. Numeri, Unitates, fractiones naturam habent Relationum. Et eatenus aliquo modo Entia appellari possunt."

[17] Leibniz [1705], II, ch.30, §4: "Les rélations ont une réalité dépendante de l'esprit commes les Verités ; mais non pas de l'esprit de l'homme, puisqu'il y a une suprême intelligence, qui les détermine toutes en tout temps."

[18] Leibniz [1710], §184; trl. Leibniz [1991], p.158. See also Leibniz [1705], II, 25, §1 and Leibniz [1875–1890], VII, p.111.

> come into being only from existing things, as I have also explained, it is necessary for eternal truths to have their existence in an absolutely or metaphysically necessary subject, that is, in God, through whom those possibilities which would otherwise be imaginary are (to use an outlandish but expressive word) realized".[19]

Leibniz even explicitly draws the conclusion that the eternal truths are invariant with respect to possible worlds:

> "And these [propositions] are of eternal truth, they will not only obtain as long as the world will remain, but they would even have obtained, if God had created the world in another way".[20]

As Robert Adams has pointed out, Leibniz' thesis that mathematical objects have their existence in God's mind might well be acceptable to a mathematical Platonist, given the necessary existence of God, given the independence of God's thought from, in particular, human thought, and given the independence of eternal truths of God's will.[21] It is therefore not surprising to see the Platonist Gödel remark in a notebook from 1944, at the time, that is, when he was studying Leibniz intensely (1943–1946), that "the ideas and eternal truths are somehow parts of God's substance", that "one cannot say that they are created by God", and that they rather "make up God's essence".[22] Gödel also writes that, of the mappings from propositions to states of affairs "the correct one" is "the one which is realized in God's mind".[23]

This aspect of Leibniz' views on mathematical objects therefore will have provided an additional interest for Gödel in a Leibnizian proof of God's existence: a corollary of such a proof for him would be that a single, fixed universe

[19] Leibniz [1875–1890], VII, pp.302–8; trl. Leibniz [1969], p.488.

[20] Leibniz [1903], p.18: "Et hae sunt aeternae veritatis, nec tantum obtinebunt, dum stabit Mundus, sed etiam obtinuissent, si Deus alia ratione Mundum creasset."

[21] Adams [1983], p.751. For Descartes, in contrast, mathematical truth is a matter of God's will, and hence on a Cartesian conception God could *choose* to make reflection true, perhaps for similar reasons as why according to Leibniz [1991], §46, Leibniz [1710], §380, God favours reflection in the physical world. See also footnote 55 below. A particularly interesting comment by Leibniz on the relation between God's will, mathematics, and creation is found in Leibniz [1695], p.57. He there says that, although irrational numbers are to some extent imperfect because they cannot be expressed as fractions, this imperfection "comes from their own essence and cannot be blamed on God"; and that, although God could have avoided creating objects (in the world) with irrational measures, if He has nevertheless done so, it is because it results in a universe with a greater variety of forms.

[22] "Die Ideen und ewigen Wahrheiten sind irgendwie Teile der göttlichen Subst[anz]. Daher kann man nicht sagen, daß sie von Gott geschöpft wurden (denn Gott wurde nicht von Gott geschöpft), sondern sie machen das Wesen Gottes aus." Gödel's Notebook *Max XI* (1944), p.31]. Compare Leibniz [1875–1890], VII, p.305, lines 1–4, which Gödel copied in a note (item 050130 in his archive), Leibniz [1710], §§335,380, and the passage in Leibniz' letter to Wedderkopf, quoted on p.23 below.

[23] "Daß eine gewisse Kombination von Begriffen oder Symbolen 'wahr' ist, bedeutet, daß sie ein adäquates Bild von etwas Existierendem ist, hängt also von der Abbildungsrelation ab. Manche Abbildungsrelationen können wir selbst konstruieren, manche (und insbesondere 'die richtigen', nämlich die im Verstand Gottes realisierten) finden wir vor", Gödel's Notebook *Phil XIV*, p.7, July 1946 or later.

of all sets V indeed exists, and hence that there is a privileged model for the axioms of set theory. Gödel describes his belief in such a privileged model in, for example, his Cantor paper from 1947.[24]

3 The reflection principle

There is an attempt of Gödel's to justify, by drawing an analogy to Leibniz' monadology, the reflection principle in set theory. Gödel never published the argument but he did present it to Wang;[25] here it will be quoted in section 4.1 below.

The basic idea behind the reflection principle is that the universe V of all sets is in some sense too large to be adequately conceivable or definable in set-theoretic terms. From this observation, one concludes to

> (1) If a clearly conceived, set-theoretical property holds of V, this property cannot be unique to V and will also characterize a set contained in it.

With respect to that property, that set is then said to "reflect" the universe.[26] (Again by reflection one then also sees that that set is not the only one to reflect the universe in that way, and that there are many more.)

Well-known applications of this informal principle are the following. The universe contains (set-theoretic encodings of) the natural numbers, hence there is also a set that contains the natural numbers (and so, by separation, there exists a set that contains nothing but the natural numbers). This use of reflection is already found in Cantor.[27] Or: of any given set, the universe contains all its subsets, hence there is also a set that contains all subsets of the given set (and so, by separation, there exists a set that contains nothing but the subsets of the given set). Or: the universe is inaccessible, hence there is an inaccessible cardinal.[28]

The first two of these applications yield justifications of two axioms of Zermelo-Fraenkel set theory, the axiom of infinity and the axiom of the powerset. Regarding the latter, note that it is not particularly clear (although for Gödel himself it apparently was) that, as the standard iterative concept of set has it,

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[24] Gödel [1990], p.181.

[25] Wang [1996], 8.7.14.

[26] E.g. Lévy [1960a], p.228, and Lévy [1960b], p.1. For two recent monographs on the reflection principle, diametrically opposed to one another in their philosophical approach, see Roth [2002] and Arrigoni [2007]. The former corresponds more closely to Gdel's view as described here.

[27] In note 2 to his paper from 1883, "On infinite, linear point manifolds 5": "Whereas, hitherto, the infinity of the first number class [...] has served as [a symbol of the Absolute], for me, precisely because I regarded that infinity as a tangible or comprehensible idea, it appeared as an utterly vanishing nothing in comparison with the abolutely infinite sequence of numbers.", Cantor [1932], p.205n.2; trl. Hallett [1984], p.42. See also Hallett [1984], pp.116–117.

[28] A cardinal κ is inaccessible if it is regular (i.e., not the supremum of k ordinals all smaller than k) and a limit (i.e., not the next cardinal greater than some cardinal λ.)

the collection of all subsets of an infinite set is a set as opposed to a proper class.[29] The informal reflection principle is a means to provide the justification needed. It is of course not excluded that alternative ways to convince ourselves of the truth of these (and other) axioms exist. Regarding the justification of the existence of inaccessibles, Gödel stated his preference for reflection over other methods in a letter to Paul Cohen of August 13, 1965:

> "As far as the axiom of the existence of inaccessibles is concerned I think I slightly overstated my view.[30] I would not say that its evidence is due *solely* to the analogy with the integers. But I do believe that a clear analogy argument[31] is much more convincing than the quasi-constructivistic argument in which we imagine ourselves to be able somehow to reach the inaccessible number. On the other hand, Levy's principle[32] might be considered more convincing than analogy".[33]

Indeed, as Wang reports, Gödel said that the justification of axioms by an appeal to reflection is the fundamental one:

> "All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now".[34]

Ackermann had stated that the notion of set is open-ended and that therefore the universe of all sets does not admit of a sharp definition (and is in that sense

[29] For Gödel's justification of the power set axiom on the iterative conception of set (not by reflection), see Wang [1974], p.174, and Wang [1996], p.220. For criticism, see e.g. Parsons [1977], p.277 and Hallett [1984], pp.236-238. Gödel's comment on an early version of Parsons [1977] seems to me to be instructive but also indicative of a weakness of Gödel's own use of idealization: "he does not understand 'idealization' broadly enough" Gödel [2003b], p.390. On a different occasion, Gödel acknowledged that there are cases where idealization is understood too broadly to be very convincing; see the quotation from his letter to Cohen that follows in the main text.

[30] Given the beginning of the preceding paragraph in the letter, "When we spoke about the power set axiom..." (p.385), presumably Gödel here refers to that same conversation.

[31] [Gödel's footnote] such as, e.g., the one obtained if an inaccessible α is defined by the fact that sums and products of fewer than α cardinals $< \alpha$ are $< \alpha$.

[32] A formulation of the idea of the unknowability of V that one also finds in Cantor and Ackermann (quoted elsewhere in this paper); in Levy's words, "the idea of the impossibility of distinguishing, by specified means, the universe from partial universes" Lévy [1960b], p.1. Levy in that paper studies four specific versions of that principle.

[33] Gödel [2003a], p.386.

[34] Wang [1996], 8.7.9. See also Wang [1996], 8.7.16.

unknowable) Ackermann [1956], p.337. (This is a reflection principle because it means that if we do find a set-theoretic property of V, this cannot be a definition of it, and hence there is a set that shares the property.) He also took this to be in accord with Cantor's intentions; and, although Ackermann does not point this out, this is indeed the principle that Cantor had used to justify the existence of the *set* of all natural numbers (see footnote 27 above).

In some sense one could say that, if Gödel's belief in this reducibility of the principles for setting up the axioms to reflection is correct, then the informal reflection principle captures the concept of set. Note that the reflection principle that Gödel has in view here is not to be confused with reflection principles that are provable in a particular formal system, such as the Montague-Levy reflection theorem in ZF.[35] By Gödel's incompleteness theorem, no single formal system for set theory can be complete, and the reflection principle Gödel is speaking about is precisely meant as the fundamental way to arrive at further axioms to extend any given system. His principle therefore has to be, and to remain, informal. Its strength increases with every application because the resulting stronger system in turn gives rise to the formulation of stronger properties to reflect.

In its fully general form (1), the principle of course cannot be upheld. For example, the property of containing every set in the universe is not reflected by any set contained in it, as such a set would have to contain itself. Reflection principles will therefore have to be precise or restrictive about the properties for which they are supposed to hold. Gödel suggested that reflection holds for *structural* properties.[36] The property of containing all sets is not structural, because it does not specify a property of all sets that might define a structure that they instantiate or exemplify. A sufficiently rich positive characterization of the notion of structural property is still wanting, but the present consideration illustrates why Gödel included it in the reflection principle that I will discuss here (the label is mine):

> (2) A *structural* property, possibly involving V, which applies only to elements of V, determines a set; or, a subclass of V thus definable is a set.[37]

Gödel's realist conception of V permits him to look for properties of V directly; this marks a deep difference with the kind of thinking about reflection that had been introduced by Zermelo.[38] Zermelo saw set theory as describing rather an open-ended, always extendable series of ever larger universes. Like Gödel, he accepted a version of the reflection principle, but, because of his different idea of what set theory is about, his principle is justified and used in a

[35] In the context of a particular formal system, the properties of V that can be reflected are of course limited by what can be expressed and defined in that system. That should contribute much to the principle's being provable, in case it is.

[36] See Wang [1977], section 3, Wang [1996], pp.283–285 and Reinhardt [1974], p.189n.1.

[37] Wang [1996], 8.7.10.

[38] Zermelo [1930].

somewhat different way.[39] According to Zermelo, V does not really exist and hence there are no literal truths to be found about it. Talk of properties of V must really be talk about the limited set of principles used in the construction of some initial segment of the open-ended series of universes.[40] This limited set of principles remains available in the construction of any longer segment of the series, and this is why the property in question will persist. In other words, we have a justification of Zermelo's reflection principle by a continuity argument.[41] Gödel, on the other hand, is not forced to construe talk of properties of V as talk about something limited; hence, reflection as exemplified by Gödel's principle has been characterized as "top-down", Zermelo's as "from below".[42] Potentially, top-down reflection is the more powerful of the two. But in its use the principle is correspondingly more difficult, as it requires one to sort out those properties of V that are not reflectable from those that are; hence Gödel's quest for "structural properties". Moreover, one might think that Zermelo's conception is to be preferred on philosophical grounds, as by accepting it, one is freed from the demands for an argument for the existence of V and for an account of how can we come to know truths about it.[43]

But it is precisely here that Gödel will have seen an advantage for his view. As Hellman, who supports and develops Zermelo's conception, has noted, that conception requires that one accepts a notion of possible objects that does not imply the existence of possibilia.[44] But as we saw in section 2, from a Leibnizian point of view such a notion of possibility cannot be accepted, and talk of possibilities that are not grounded in something existent is ultimately unintelligible. The same criticism would be applied to any other interpretation of set theory in which commitment to the existence of V is avoided by resorting to modal notions.[45] According to Gödel, the open-endedness of the notion of set that motivates resorting to notions of possibility is not the correlate of an ontological fact: "To say that the universe of all sets is an unfinishable totality does not mean objective undeterminedness, but merely a subjective inability to

[39] On their differences, see also the extensive discussion (from a somewhat different perspective) by Tait [1998].

[40] "Construction" in the sense that the existence of this segment is derived from specific axioms by specific principles. In classical set theory, such axioms and principles will themselves generally not be "constructive" in the sense in which that term is used to characterize varieties of mathematics such as intuitionism.

[41] The logic of open-ended series is intuitionistic rather than classical. This type of reasoning we will see again later on in this paper, see footnote 83. For more on this type of argument and its justifications, see Van Atten & Van Dalen [2002]. Georg Kreisel wrote to me in a letter of March 7, 2006, that in the period that he knew Gödel, the latter was "sympathetic to a justification by intuitionistic logic (in terms of not necessarily constructive knowledge)" of set-theoretic reflection principles.

[42] E.g., Hellman [1989], p.90.

[43] From the point of view of constructive mathematics in the sense explained in footnote 40, what remains to be accounted for in Zermelo's conception would of course still be far too much.

[44] Hellman [1989], pp.57,58.

[45] Yourgrau's criticism of Parsons' position is of the same type. See Parsons [1977], pp.268–297 and Yourgrau [1999], p.177–185.

finish it".[46] (Here, "subjective" seems to refer to the act, however idealized, of obtaining a collection by putting it together from elements which are considered to be given prior to that act. Cantor's notion of set (quoted above) contains a subjective element in just this particular sense. The universe V can never be obtained in such an act, as V cannot be a set.) A closely related Leibnizian observation is made by Mugnai:

> "In man's limited intellect there is a distinction between the 'capacity to think' and the 'actual exercise' of that capacity. This distinction is not met within God. If the ideas *in Mente Dei* are conceived as 'dispositional properties' then we must also postulate a 'state' of the divine intellect in which it carries out a limited activity, during which all the totality of ideas are never present all at once. This is surely unacceptable from the theological point of view, however, since it limits the divine powers and assimilates the psychological and reasoning activity of God to the example of human activity".[47]

4 Gödel's analogy argument for the reflection principle

4.1 Presentation of the argument

Gödel's argument for principle (2) that I should like to analyze (not his only one) consists in drawing an analogy to Leibniz' monadology. Here I will present that argument, try to fill in the details, consider the question whether it is a good argument, and conclude that it is not. In doing so, I will not be arguing that the alternative arguments that Gödel had for the validity of reflection principles are incompatible with a Leibnizian metaphysics. What I am going to argue is that the one argument we know of in which Gödel explicitly tries to argue from a Leibnizian metaphysics to a form of the reflection principle in set theory does not work.

A note on the sources that will be used here: as yet, Gödel's philosophical notebooks have been transcribed only partially. For all I know there may be material in those untranscribed parts that is relevant to the matter at hand. As a principle of interpretation, I will assume that the argument that Gödel presented to Wang in the 1970s, when he had perfect access to his notebooks from the 1940s (except for the one from 1945–1946 that he reported lost), is the version that he considered best. As for Leibniz, I have tried to use, whenever possible, writings from 1686 and later, as that is the phase in the development

[46] Wang [1996], 8.3.4. Tait ((1998), p.478) wishes to leave open the same possibility of objective undeterminedness that Gödel denies.

[47] Mugnai [1992], p.24. Similarly, Jolley ((1990), p.138) notes: "Now Leibniz might be more reluctant than Mates to allow that divine ideas are dispositions, for this may be difficult to reconcile with the traditional view that God is pure act."

of Leibniz' philosophy that in 1714 culminated in the Monadology. But in particular cases earlier texts may be relevant as well.

What might motivate one to draw an analogy between monadology and set theory is that in both cases we have a universe of objects, the objects resemble in some sense the whole, and the actual universe is in some sense the best out of a collection of possible universes. In the monadology, God chooses a universe or world to actualize from out of the collection of possible worlds, according to some criteria for which one is best; in set theory, models for ZFC are known which are generally not believed to correspond to set-theoretical reality (e.g., the so-called "minimal model" is considered not to be the "best" model because it is too small). The themes of reflection and mirroring occur often in Leibniz' writings. A typical example is Leibniz' formulation of his Principle of Harmony in section 56 of the *Monadology*:[48]

> "Each simple substance has relations that express all the others, and is in consequence a perpetual living mirror of the universe".[49]

One could use the monadology as a means to generate structural principles for monads and their relations, substitute in such a principle the notion of set for that of monad, and then seek independent reasons why the set-theoretical principle thus obtained should be true. The justification one might then come up with will not depend on an analogy between the universes of monads and sets. This merely heuristic approach was followed by Joel Friedman in his paper "On some relations between Leibniz' monadology and transfinite set theory"[50] where he obtained maximizing principles in set theory on the basis of maximizing principles of harmony in the monadology. A similar somewhat loose (but not necessarily less fruitful) approach was taken by Wim Mielants in his paper "Believing in strongly compact cardinals", where "Leibniz's philosophy is only a source of inspiration for the maximization properties we use here".[51] One conclusion that may be drawn from the present paper is that such a heuristic approach will probably be more fruitful than an analogy of the type Gödel wished to draw.

Gödel's analogy is one that he takes to be by itself a justification of a form of the reflection principle, without the need to adduce independent reasons. As will be discussed later, a convincing analogy argument does not always require that the situation with which an analogy is drawn is, in its full extent, actual or real. But, to look ahead a bit, Gödel's use of his analogy as a sufficient justification is based on the idea that the reflection principle is true in set theory for exactly the same reason why a certain monadological proposition is true. As long as it is not clear that such a general reason, should it exist at all, would involve no specifically monadological notions, it is not clear whether here, too, justification can be treated independently of a justification of the monadology. For the moment I will leave it an open question whether one has to accept

[48] The actual name is given to it in section 78.
[49] Leibniz [1991], section 56.
[50] Friedman [1975].
[51] Mielants [2000], p.290.

the monadology as the true metaphysics in order to be convinced by Gödel's argument, and concentrate rather on the prior task of filling in the details of the analogy that he indicates.

Hao Wang recorded Gödel's argument in item 8.7.14 of his *Logical Journey.* For clarity, I quote the preceding item as well:

> "8.7.13 [...] Consider a property $P(\mathrm{V}, x)$, which involves V. If, as we believe, V is extremely large, then x must appear in an early segment of V and cannot have any relation to much later segments of V. Hence, within $P(\mathrm{V}, x)$, V can be replaced by some set in every context. In short, if P does not involve V, there is no problem; if it does, then closeness to each x helps to eliminate V, provided chaos does not prevail."
>
> "8.7.14 There is also a theological approach, according to which V corresponds to the whole physical world, and the closeness aspect to what lies within the monad and in between the monads. According to the principles of rationality,[52] sufficient reason, and preestablished harmony, the property $P(\mathrm{V}, x)$ of a monad x is equivalent to some *intrinsic* property of x, in which the world does not occur. In other words, when we move from monads to sets, there is some set y to which x bears intrinsically the same relation as it does to V. Hence, there is a property $Q(x)$, not involving V, which is equivalent to $P(\mathrm{V}, x)$. According to medieval ideas, properties containing V or the world would not be in the essence of any set or monad".[53]

So in the case for sets, the claim is that $P(\mathrm{V}, x) \equiv Q(x)$, where $Q(x) = \exists y P(y, x)$ and x and y are sets. (Certainly, the fact that $Q(x)$ is a one-place predicate does not suffice to make it express a non-relational property.[54])

The approach is "theological" because in the monadological setting, it is a central monad or God who creates a universe of objects.[55] To make Gödel's analogy more explicit, I propose to put it in a slightly different form, the rationale of which will be explained as we go along. As Gödel adds the explanation that "according to medieval ideas, properties containing V or the world would

[52] By this, I take it, Gödel means the principle of contradiction.

[53] Wang [1996], 8.7.14.

[54] See Ishiguro [1990], ch.6.

[55] A curious example of a theological approach by Gödel to a mathematical question is found in his notebook *Max X* (1943–1944), p.18: "Does the commandment that one shall make neither likeness nor image perhaps also mean, that type theory must be accepted and that any formalisation of the all leads to a contradiction?" ("Bedeutet vielleicht das Gebot, du sollst dir kein Gleichnis noch Bildnis machen, auch, daß die Typentheorie anzunehmen ist und jede Formalisierung des Alls zu einem Widerspruch führt?"). The inference from a commandment to a mathematical truth would seem to fit a Cartesian view of the relation between God and mathematics better than a Leibnizian one. For Descartes, mathematical truth was determined by God's will; Leibniz contested this. For an analysis of this difference between Descartes and Leibniz, see Devillairs [1998]. More positive statements by Gödel on type-free logic occur in, for example, his correspondence with Gotthard Günter, see Gödel [2003a], pp.527,535.

not be in the essence of any set or monad", it is clear that he in this analogy argument considers only essential properties. He first presents, in effect, the following monadological proposition:

> "Essential, relational properties of (created) monads are intrinsic properties in which the universe as a whole does not occur but part of it does."

"Part" here is meant in the proper sense according to which no part of the universe expresses the whole universe perfectly; this is in fact implied by the condition that in the properties in question "the universe as whole does not occur". The notion of expression Leibniz describes as follows:

> "That is said to express a thing in which there are relations which correspond to the relations of the thing expressed".[56]

> "It is sufficient for the expression of one thing in another that there should be a certain constant relational law, by which particulars in the one can be referred to corresponding particulars in the other".[57]

> "One thing expresses another (in my terminology) when there exists a constant and fixed relationship between what can be said of one and of the other".[58]

Clearly, a perfect expression of x by y requires a 1-1 correspondence between all properties of x and (some) properties of y.

Let us call the above monadological proposition the "reflection principle for (created) monads". Gödel then proposes that we move from monads to sets and obtain from this, by analogy, the reflection principle for sets:

> "Essential, relational properties of sets are intrinsic properties in which V does not occur but a set does."

In the move from monads to sets, the immediate analogue of a part (in the strong sense) of the universe of monads (a collection of monads) is a part of the universe of sets, hence a collection of sets and not an individual set. But this actually suffices, because of the following principle that Gödel accepted: any collection that is properly contained in V and that cannot be mapped 1-1 to it (and in that sense cannot perfectly "express" V), is not a proper class but a set. This is known as "Von Neumann's axiom".[59] So although the immediate analogue of a collection of monads that does not perfectly express the universe of monads is a collection of sets that does not perfectly express V, by Von

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[56] Leibniz [1969], p.207.
[57] Leibniz [1903], p.15; trl. Rutherford [1995], p.38.
[58] Leibniz [1875–1890], II, p.112; trl. Mates [1986], p.38n.11.
[59] The idea had already been formulated by Cantor in a letter to Dedekind of July 28, 1899, first published in Cantor [1932], seven years after Von Neumann's paper (1925). For a clear and detailed discussion of this axiom, see Hallett [1984], section 8.3.

Neumann's axiom the latter collection is itself a set. It is this set the existence of which Gödel's analogy argument concludes to.

Gödel commented on Von Neumann's axiom:

> "As has been shown by Von Neumann, a multitude is a set if and only if it is smaller than the universe of all sets.[60] The great interest which this axiom has lies in the fact that it is a maximum principle, somewhat similar to Hilbert's axiom of completeness in geometry. For, roughly speaking, it says that any set which does not, in a certain well defined way, imply an inconsistency exists".[61]

This fits well into Leibniz's picture according to which mathematical existence is equivalent to mathematical possibility, and the latter is wholly determined by a (global) principle of non-contradiction; we will come back to this later.

4.2 The analogy is ineffective

The conception of analogy arguments I will use here is Kant's, who in section 58 of the *Prolegomena* writes: "Such a cognition is one by analogy, which does not signify for example, as the word is commonly understood, an imperfect similarity of two things, but a perfect similarity of two relations between entirely dissimilar things."[62] If the similarity in question is perfect, it will be embodied in a general principle that governs both of the domains involved in the analogy. Only the existence of such an underlying general principle can give an analogy argument genuine force. Of course, once such a general principle has been identified, it can be used to construct a direct argument for the desired conclusion, and the analogy is no longer necessary. The function of the analogy will then have been to have pointed to the relevant general principle.[63]

So in order to show that the similarity claimed by Gödel is not arbitrary or superficial, but does indeed carry argumentative weight, it would have to be shown that the reflection principle holds for monads because they instantiate a more general principle that implies reflection for universes of objects satisfying certain conditions. Applying that same more general principle to the universe of sets should then yield the reflection principle for sets.[64]

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[60] Wang [1996], 8.3.7.

[61] Wang [1996], 8.3.8. The inconsistency Gödel refers to here is the inconsistency arising from conceiving of a particular kind multitude as set. As we saw above, for Gödel V genuinely exists, but as a mere multitude and not as a set.

[62] "Eine solche Erkenntnis ist die nach der Analogie, welche nicht etwa, wie man das Wort gemeiniglich nimmt, eine unvollkommene Ähnlichkeit zweier Dinge, sondern eine vollkommene Ähnlichkeit zweier Verhältnisse zwischen ganz unähnlichen Dingen bedeutet." Gödel will surely have known this passage; but in his copy of the Reclam 1888 edition of the *Prolegomena*, there are no reading marks to it. (I am grateful to Marcia Tucker at the Historical Studies-Social Science Library of the IAS for having verified this.)

[63] To emphasize that this is the function of an analogy, St. Augustine classified it with the signs (Maurer [1973]).

[64] In his formulation of reflection principle (2) on p.9 above, Gödel mentions a restriction on the properties that can be reflected, saying that they should be "structural". I will come back to the possible role of this restriction in the analogy later.

But such a principle, I claim, cannot exist. In a first step, I argue that it is consistent with the purely metaphysical principles of the monadology to assume that the reflection principle for monads holds but the reflection principle for sets fails. In the second step, I explain why this entails that Gödel's analogy is ineffective, whether the monadology is true or not.

That in the monadology the reflection principle for monads is consistent follows from the fact that, as I will now argue, in the monadology that principle is true.

As a preliminary, the meaning of the term "essence" has to be clarified. Leibniz uses it in different ways. Sometimes he defines the essence of a monad as simply the collection of all its properties, considered in abstraction from the existence of that monad. As he holds that each monad expresses the whole universe or world, by this definition it is trivially false that the essence of a monad does not involve the world.[65] But Leibniz also has another notion of essence, which is the one that will be relevant here. This notion is defined as the collection of all the necessary properties of that substance. For example, in 1676 Leibniz first defines an "attribute" as "a necessary predicate conceived through itself, or that cannot be analysed into several others" and then "an *essence* is [...] the aggregate of all the attributes (of a thing)".[66] In 1678 he defines the "essence of a thing" as "the specific reason of its possibility" and specifies that what is true in the region of essences is "unconditionally, absolutely and purely true".[67] This definition he repeats two decades later, in 1701, "the essence of the thing being nothing but that which makes its possibility in particular".[68] Of particular interest for its idealistic content is Leibniz' remark in the *New Essays* (1705) that possibility is the same as being distinctly intelligible (which intelligibility is ruled out for contingent properties).[69] Finally, in 1714, he writes that

[65] While reading Leibniz [1903], Gödel noted: "The proposition that every thing involves all others, can be understood purely logically. Namely: It involves all accidents, among these however also the relations to all other things; these however involve the other things. But that is only an accidental, no necessary involvement. But to the extent that to the essence belongs the reaction in arbitrary situations, it also involves essentially—also through knowledge (mirror)—accidentental involvement." ("Die Aussage, daß jedes Ding alle andere involviert, kann rein logisch verstanden werden. Nämlich: Es involviert alle Acc[identia], unter diesen aber auch die Beziehungen zu allen anderen Dingen; diese involvieren aber die anderen Dinge. Das ist aber nur ein accident[elles], kein notwendiges Involvieren. Aber insofern zum Wesen die Reaktion in beliebigen Lagen gehört, involviert [es?] sie auch essentiell—auch durch Erkenntnis (Spiegel)— acci[dentelles] Involvieren.") Gödel's Notebook *Max X* (1943–1944), pp.70–71. Here Gödel must be referring to Leibniz' statement on p.521 of that edition, "Every singular substance involves in its perfect notion the whole universe" ("Omnis substantia singularis in perfecta notione sua involvit totum universum").

[66] Leibniz [1923–], VI, iii, p.574, as quoted in Adams [1994], p.127.

[67] Leibniz [1923–], II, i, pp.390 and 392, as quoted in Adams [1994], pp.136,138.

[68] Leibniz [1875–1890], IV, p.406: "l'essence de la chose n'étant que ce qui fait sa possibilité en particulier".

[69] Leibniz [1875–1890], V, p.246: "But whether they depend on the mind or not, it suffices for the reality of their ideas, that these modes are *possible* or, which is the same thing, distinctly intelligible." ("Mais soit qu'ils dependent ou ne dependent point de l'esprit, il suffit pour la realité de leur idées, que ces Modes soyent possibles ou, ce qui est la même chose, intelligibles distinctement.") Gödel noted this one, see item 050131 in his archive.

> "I consider possible everything that is perfectly conceivable, and which therefore has an essence, an idea; without taking into consideration whether the other things allow for it to come into being".[70]

With this notion of essence in place, the argument for Reflection for created monads proceeds as follows:

1. All properties of monads consist in their own perceptions; this does not rule out relational properties as these are intrinsic too. (Premise)
2. Essential properties correspond to distinct perceptions. (Premise)
3. No created monad can distinctly perceive the whole universe. (Premise)
4. Essential, relational properties of (created) monads are intrinsic properties in which the universe as a whole does not occur but part of it does. (From 1, 2 and 3)

In the opening sections of the *Monadology*, Leibniz says that monads are the ultimate constituents of reality. They are simple in the sense that they are not composed out of parts (section 1). Elsewhere, Leibniz also says that the monads are not in space and time, but that space and time are rather phenomena that depend on the way monads represent reality to themselves. Although monads are simple, they do have inner states, and these can change. This does not contradict the fact that they have no parts, if this is understood to mean (in terms of Husserl's third *logische Untersuchung*) that they have no independent parts but only dependent ones, like a continuum.[71] The changes arise within the monad itself and do not come from outside, for monads have no parts that can be acted upon from outside; they "have no windows" (section 7). Only God can be said to act upon the created monads directly. Leibniz identifies the specification and variety of simple substances with the internal complexity of these inner states (section 12), and calls these transitory states "perceptions" (section 14). The properties of a monad consist in its proper perceptions.[72] Perceptions "enfold and represent a multiplicity in a unity, or in the simple substance" (section 14), and in fact each monad perceives or represents the whole universe.[73] Various crucial points for Gödel's analogy are now made in section 60:

[70] To Bourguet, December 1714, Leibniz [1875–1890], III, pp.573–574: "J'appele possible tout ce qui est parfaitement concevable, et qui a par consequent une essence, une idée : sans considerer, si le reste des choses luy permet de devenir existant." See also Leibniz [1991], section 43, Leibniz [1710], section 390.

[71] Leibniz used the absence of independent parts as an argument against the conception of the mind as a machine or mechanism: the mind is a unity, whereas a machine has (independent) parts, e.g. in his *New System of the Nature and Communication of Substances* from 1695, Leibniz [1969], p.456. Gödel appealed to the very same argument: "Consciousness is connected with one unity. A machine is composed of parts", Wang [1996], 6.1.21.

[72] The special case of reflexive knowledge or consciousness that some monads sometimes have of their inner states, apperception, plays no role in Gödel's analogy.

[73] Compare also the earlier *On Nature's Secrets* from around 1690: "Indeed, the multiple finite substances are nothing other than diverse expressions of the same universe according to diverse respects and each with its own limitations." Leibniz [1875–1890], VII, p.311n., trl. Leibniz [1991], p.217.

> "For in regulating the whole, God has had regard for each part, and in particular for each monad, which, its very nature being representative, is such that nothing can restrict it to representing only part of things. To be sure, this representation is only confused regarding the detail of the whole universe. It can only be distinct in regard to a small part of things, namely those that are nearest or most extensively related to each monad. Otherwise each monad would be a deity. It is not in their object [namely the whole universe], but in the particular mode of knowledge of this object that the monads are restricted. They all reach confusedly to the infinite, to the whole; but they are limited and differentiated by the degrees of their distinct perceptions."[74]

If monads did not differ this way, they would all be one and and the same, by identity of indiscernables (which is a consequence of Sufficient Reason). For the only properties monads have are perceptual, and perceptions differ only in degree of distinctness.[75] Only the monad which is God perceives the whole universe perfectly; the perception of the universe by created monads necessarily is (partly) confused, because their receptivity is necessarily limited (secton 47).[76] It follows that the perceptions of no created monad can exhaust the universe. This precludes that the perceptions of a created monad stand in 1-1 relation to the elements of the universe, and therefore no created monad expresses the universe perfectly.

Note in passing how the fact that monads have no windows and only God acts directly upon them explains, when combined with the idea that sets are objects in God's mind, Gödel's assertion to Paul Benacerraf that the monads have unambiguous access to the full set-theoretic hierarchy (see footnote 3 above; this part of the anecdote is also reported in Maddy [1990], p.79). As Leibniz wrote around 1712:

> "I am convinced that God is the only immediate external object of souls, since there is nothing except him outside of the soul which acts immediately upon it. Our thoughts with all that is in us, in so far as it includes some perfection, are produced without interruption by his continuous operation. So, inasmuch as we receive our finite perfections from his which are infinite, we are immediately affected by them. And it is thus that our mind is affected immediately by the eternal ideas which are in God, since our mind has thoughts which are in correspondence with them and participate in them. It is in this sense that we can say that our mind sees all things in God."[77]

[74] Leibniz [1991], section 60.

[75] Gödel writes in his Notebook *Max X* (1943–1944), p.20: "Almost any property can be had to different degrees" ("Man kann fast alle Eigenschaften in verschiedenen Graden haben").

[76] Necessarily, for by identity of indiscernables God is unique; section 39 cites, alternatively, the principle of sufficient reason.

[77] "Conversation of Philarète and Ariste" (one of the direct forerunners of the *Monadology*), Leibniz [1875–1890], VI, p.593; trl. Leibniz [1969], p.627. Gödel seems to have had this or

The fact that all of a monad's properties are internal to it might seem to rule out relational properties, in which case Gödel's analogy argument would not work, for if there are no relations between monads then there is no basis for an analogy concluding to the existence of relations between sets. In fact, on Leibniz' understanding of relations, relational properties are not at all ruled out: a monad x will have a relational property P if x expresses the relata in the way characteristic for P. But to express other monads this way is an entirely internal property; it does by itself not guarantee that these other monads indeed exist. This is indeed what Leibniz meant, as he makes clear in his reply to an objection made by his correspondent Des Bosses. Des Bosses had written to Leibniz:

> "If the monads of the universe get their perceptions out of their own store, so to speak, and without any physical influence of one upon the other; if, furthermore, the perceptions of each monad correspond exactly to the rest of the monads which God has already created, and to the perceptions of these monads, and are harmonized so as to represent them; it follows that God could not have created any one of these monads which thus exist without constructing all the others which equally exist now, for God can by no means bring it about that the natural perception and representation of the monads should be in error; their perception would be in error, however, if it were applied to nonexistent monads as if they existed."[78]

And Leibniz replied:

> " He can do it absolutely [i.e., as far as logic is concerned]; he cannot do it hypothetically [i.e., when also God's will is taken into account], because he has decreed that all things should function most wisely and harmoniously. There would be no deception of rational creatures, however, even if everything outside of them did not correspond exactly to their experiences, or indeed if nothing did, just as if there were only one mind; because everything would happen just as if all other things existed, and this mind, acting with reason, would not charge itself with any fault. For this is not to err. [...] Not from necessity, therefore, but by the wisdom of God does it happen that judgements formed upon the best appearances, and after full discussion, are true."[79]

So in what Leibniz calls an "absolute" sense, a monad can have a relational property without that relation obtaining in the world. But in the actually cre-

a similar passage, e.g. Leibniz [1686], section 28, in mind when he remarked in his letter to Gotthard Günther of April 4, 1957: "That abstract conceptual thought enters individual monads only through the central monad is a truly Leibnizian thought.", Gödel [2003a], p.527.

[78] Des Bosses to Leibniz, April 6, 1715, Leibniz [1875–1890], II, p.493; trl. Leibniz [1969], p.611.

[79] Leibniz to Des Bosses, April 29, 1715, Leibniz [1875–1890], II, p.496; trl. Leibniz [1969], p.611. See also Leibniz [1686], section 14, Leibniz [1710], section 37, and Leibniz [1875–1890], IV, p.530.

ated world this is excluded, for in choosing that world God sees to it that the perceptions of its monads are in harmony with one another.[80] This depends on God's will instead of logic and that is why Leibniz says that it is not "absolutely" but "hypothetically" necessary that relational properties express relations that indeed obtain. In the presence of this principle of harmony, the circumstance that a monad x in a world truly has relational property P not only implies, but is equivalent to, the circumstance that it has an appropriate intrinsic property. This explains why Gödel mentions the principle harmony in his analogy argument: as he wishes to reason by analogy that there exists a set y that is related to the set x by $P(y, x)$, he needs, in the domain to which the analogy is drawn, the existence of a monad (or collection of monads; see below) y for the monad x to relate to. Without a principle of harmony, that existence would not be guaranteed.

The following step is to see that, more specifically, properties that are essential correspond to perceptions that are distinct. Leibniz understands by necessary properties those that admit of finite analysis into primitive ones (section 33). They cannot involve confused perceptions, as those combine many perceptions into one in such a way that there is no complete, finite analysis into distinct perceptions. In the *Monadology*'s twin, the paper *Principles of Nature and Grace* from the same year, 1714, Leibniz states in section 13 that "Our confused perceptions are the result of the impressions which the whole universe makes upon us".[81] They therefore correspond to, or express, contingent truths (*Monadology* section 36). God knows contingent truths a priori, but not by demonstration. An infinite demonstration is impossible according Leibniz, as such an object would form an infinite whole, which he believed could not exist; rather, God knows contingent truths by a (direct) "infallible vision".[82] There is a continuum of qualities of perception, of which complete distinctness is one extreme. The more distinct a perception is, the more it contributes to the individuality of a monad, to the point where complete distinctness corresponds to essential properties.

In particular, a relational property of a monad that is part of its essence demands that its expression of all relata is clear and distinct. It follows that, as Gödel says, it cannot be an essential property of any monad x to stand in a relation P to the universe. A monad may well stand in a relation P to the universe but this will then not be an essential property of the monad. Suppose that one finds a necessarily true proposition A that says of a created monad x that it stands in a relation P to the universe. For Leibniz, that A is a necessary truth means that A expresses an essential property of x. For the reason just given, what specifically makes A true cannot involve the whole universe but only a proper part of it. Hence, A is equivalent to a proposition B that says

[80] Also: "It can be said that God arranges a real connection by virtue of that general concept of substances which implies perfect interrelated expressions between all of them, though this connection is not immediate, being based on what God has wrought in creating them.", Leibniz [1875–1890], II, pp.95–96; trl. Rutherford [1995], p.146.

[81] Leibniz [1875–1890], VI, p.604.

[82] "On Freedom" (1689), Leibniz [1973], p.111.

that x stands in a necessary relation P to just part of the universe. By the principle of harmony, a part of the universe such as perceived by x indeed exists. Thus we have arrived at what we have called a 'reflection principle for (created) monads'.[83] As noted above (p.14), this argument does not yield the conclusion that there is a monad to which x is related, but that there is a part of the universe (in the sense of a collection of monads that does not express the universe perfectly) to which it is related; we also saw why, in the presence of Von Neumann's axiom, this suffices for Gödel's analogy. If an individual monad z such that x stands in the same relation to z as it does to the collection of monads y is possible, then it could be argued that God would go on actually to create that monad z, on the ground of a principle of maximality or plenitude (which is a form of the principle of harmony).[84]

At this point, the following might seem to be a quick argument against Gödel's analogy. Reflection for monads depends on God's will (namely, on His choice to create a universe that is harmonious), and is in that sense contingent; reflection for sets, on the other hand, is supposed to be a necessary principle. But then these two forms of reflection cannot be true on the ground that both instantiate one and the same general principle. However, this argument does not succeed, because the general principle might be (or could be made) conditional on harmony: 'For all harmonious universes, ...'. In the case at hand, all that harmony amounts to is the requirement that, if an object in a universe has a relational property, the relata also exist in that universe. For the universe of monads this needs, as we saw, some argument, while for the universe of sets it seems trivial. But for the applicability of the general principle the reason why a universe is harmonious would not matter, only that it is.

Instead, the argument against Gödel's analogy proceeds from the fact that, in contrast to reflection for monads, it is consistent with the monadology that no reflection principle for sets holds. This is because the monadology poses no metaphysical constraints on the essential mathematical properties that a set (or any object of pure mathematics) can have. The explanation for this is as

[83]This principle is of course closely related to the ancient and medieval idea that things are known according to the capacities of the knower, and that hence a lower being's knowledge of a higher being is necessarily incomplete. A difference between that idea and reflection is that only the latter explicitly concludes to the existence of a third object (with a certain property). But in a formulation of Odo Reginaldus from around 1243–1245, that conclusion is more or less present: 'How can a finite being reach the infinite? About this, some others have said that God will present himself to us moderated, and that he will show himself not in his essence, but in a creature', Côté [2002], p.78, trl. mine. ("Quomodo potest finitum attingere ad infinitum? Propter hoc dixerunt alii quod deus contemperatum se exhibebit nobis, et quod ostendet se nobis non in sua essentia, sed in creatura."—Odo then comments that this opinion has fallen from favour ("Sed hec opinio recessit ab aula"), which, theologically, is not surprising.) From here it is only a small step to: 'Suppose creature A has a perception of God. Then God is capable of making a creature B such that A's perception cannot distinguish between God and B.' The argumentation here is reminiscent of continuity arguments. Côté's monograph (2002) is an invaluable analysis of the medieval discussion of finite beings' knowledge of an infinite God.

[84]Leibniz [1923–], VI, iii, 472: 'After due consideration I take as a principle the Harmony of things, that is, that the greatest amount of essence that can exist does exist.' (1671), trl. Mercer [2001], pp.413–414.

follows.

As we saw in section 2, the objects of pure mathematics are, for Leibniz, entirely mental objects, and have their primary and original existence in God's mind. As a consequence, the existence of relations between pure sets or collections (in particular, V) will have no foundation in a created monad. Relations between pure sets or collections are, ontologically, relations between God and himself. Relations have their ultimate reality in God's being able to think them. But, contrary to the case of created substances, for Leibniz there are no intrinsic limitations to God's thinking other than non-contradiction. 'Possible things are those which do not imply a contradiction', he says,[85] and God thinks all possibilities:

> "The infinity of possibles, however large it may be, is not larger than that of the wisdom of God, who knows all possibles".[86]

What is true in mathematics, and in particular what relations can obtain between mathematical objects, depends only on the Principle of Contradiction. The Principle of Sufficient Reason and its consequences have no influence on what is or is not the case in mathematics. Leibniz explains this in his second letter to Clarke, from 1715:

> "The great foundation of mathematics is the *principle of contradiction or identity*, that is, that a proposition cannot be true and false at the same time and that therefore A is A and cannot be non-A. This single principle is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles. But in order to proceed from mathematics to natural philosophy, another principle is requisite, as I have observed in my *Theodicy*; I mean the *principle of a sufficient reason* [...] By that principle, viz., that there ought to be a sufficient reason why things should be so and not otherwise, one may demonstrate the being of a God and all the other parts of metaphysics or natural theology".[87]

Leibniz says this to support his contention that the mathematical principles of the materialist philosophers are the same as those of Christian mathematicians, the difference between them rather being the metaphysical one that the Christians admit immaterial substances. As Leibniz sees it, the truths of metaphysics (i.e., the principles specifically about monads and their relations to one another) all follow from the principle of sufficient reason (together with the principle of contradiction), but the principle of contradiction is prior to the principle of sufficient reason. In particular, then, sufficient reason and its consequences are compatible with any relation that obtains between pure possibilities. As a

[85] Leibniz to Joh. Bernoulli, February 21, 1699, Leibniz [1849–1863], III, p.574: 'Possibilia sunt quae non implicant contradictionem'.

[86] Leibniz [1710], section 225: "L'infinité des possibles, quelque grande qu'elle soit, ne l'est pas plus que celle de la sagesse de Dieu, qui connaît tous les possibles."

[87] Leibniz [1969], pp.677–678. Also section 9 in the fifth letter to Clarke Leibniz [1969], p.697, and Leibniz [1710], section 351.

special case, no metaphysical principle constrains what is true in pure mathematics. This idea one finds in both early and late Leibniz. For example, the young Leibniz wrote to Magnus Wedderkopf (May 1671),

> "No reason can be given for the ratio of 2 and 4 being the same as that of 4 and 8, not even in the divine will. This depends on the essence itself, or the idea of things. For the essences of things are numbers, as it were, and contain the possibility of beings which God does not make as he does existence, since these possibilities or ideas of things coincide rather with God himself".[88]

And, much later, in a letter to Pierre Varignon of June 20, 1702,

> "Entre nous je crois que Mons. de Fontenelle, qui a l'esprit galant et beau, en a voulu railler, lorsqu'il a dit qu'il vouloit faire des elemens metaphysiques de nostre calcul".[89]

As Michel Fichant has concluded,

> "The idea of a metaphysics of the calculus of the infinite, or of a metaphysical transposition of a consideration on the calculus of the infinite, is entirely alien to Leibniz; whenever someone ventured in that area, he has always objected to it".[90]

This absence of a metaphysical constraint on mathematical truth implies that no description or reasoning in purely metaphysical terms can lead us to the discovery of an underlying general principle that would imply that reflection holds for sets, too, as such a metaphysical description will be equally compatible with the falsehood of reflection for sets. Yet, Gödel's analogy argument in effect precisely attempts to draw attention to a general principle in this way. Gödel first describes a purely metaphysical fact, namely the reflection principle for monads, and then arrives at the desired mathematical conclusion by, as he says, "moving from monads to sets". This analogy argument therefore fails. Of the reason for this failure, i.e. the fact that metaphysical principles do not

[88] Leibniz [1969], p.146.

[89] Leibniz [1849–1863], IV, p.110.

[90] Fichant [2006], pp.29–30: "L'idée d'une métaphysique du calcul de l'infini ou d'une transposition métaphysique d'une réflexion sur le calcul de l'infini est totalement étrangère à Leibniz ; il l'a toujours récusée chaque fois que quelqu'un s'est aventuré dans ces parages". A few lines further on, he writes: "It is true that he says in a famous letter to Varignon that 'the real never fails to be perfectly governed by the ideal and the abstract', on account of which, in effect, mathematical calculations are applicable to nature, but, basically, to nature inasmuch as the real in question is at the level of the phenomena, not at that of the substances." ("Il est vrai qu'il dit dans une lettre célèbre à Varignon, que 'le réel ne laisse pas de se gouverner parfaitement par l'idéal et l'abstrait', ce qui fait que, effectivement, les calculs mathématiques sont applicables à la nature, mais, au fond, à la nature pour autant que le réel dont il est alors question se situe sur le plan du phénomène, et non sur celui des substances.") A curious exception to Leibniz's advocated practice of keeping metaphysical principles out of purely mathematical arguments occurs in his attempts to show that absolute space is Euclidian, which appeal to the principle of sufficient reason. For a full discussion of this exception, see De Risi [2007], pp.252–264.

constrain pure possibilities,[91] two further consequences should be noted. First, adding, in particular, a metaphysical principle that somehow corresponds to the restriction in Gödel's formulation of reflection principle (2) on p.9 above that the set-theoretical properties to be reflected should be *structural* will not help making the analogy work. A second consequence is that the monadology will not suggest a *disanalogy* with the reflection principle for sets either. No truth about monads and their relations can contradict mathematics, for on Leibniz' conception, God's acts of creation are (voluntary) acts of applying mathematics.[92]

The argument against Gödel's analogy does not depend on Leibniz' specific construal of mathematical possibility in terms of non-contradiction; what it depends on is the more general condition that the notion of pure possibility that defines mathematical truth is a boundary condition on the possible worlds out of which the metaphysical principles select one. Any notion of mathematical possibility that guarantees invariance of mathematical truth with respect to possible worlds will satisfy this condition.[93]

Generally speaking, a successful analogy from a state of affairs in one domain to a state of affairs in another may or may not presuppose that the first domain actually exists. The only function of the description of a state of affairs in the

[91] One of Leibniz' manuscripts of around the time of the *Monadology* is titled "The metaphysical foundations of mathematics", Leibniz [1969], pp.666–674. But in it, Leibniz actually proceeds by defining and developing pure concepts; it is metaphysical on account of the great generality of them. An example is his argument for the proposition that the whole is always greater than the part. But there is no mention whatsoever of monads and the principles governing them and their relations, and therefore not metaphysical in that sense.

[92] "God makes the world while calculating and exercising knowledge" ("Cum Deus calculat et cogitationem exercet, fit mundus"), Leibniz [1875–1890], VII, p.191n, and "Necessity in geometry is absolute, but it follows that this is also the case in physics, because the supreme Wisdom, who is the source of things, acts as the most perfect geometer and observes harmony" ("Absolutae est necessitatis in Geometria, sed tamen succedit et in Physica, quoniam suprema Sapientia, quae fons est rerum, perfectissimum Geometram agit et Harmoniam observat"), Leibniz [1849–1863], VI, p.129. (The notion of absoluteness here must be a wider one than the one in Leibniz' letter to Des Bosses, quoted on p.19 above, from which harmony is explicitly excluded.) For general discussion of this idea, see Osterheld-Koepke [1984], pp.138–144. The idea also contributes to an explanation of the following observation by Gödel from 1942, Notebook *Max VI*, p.380: "The principle that every math[ematical? metaphysical?] proposition has a generalisation for arbitrary higher cardinality (but not the other way around) expresses one of the most general properties of the structure of the world. Namely: Everything is mirrored in everything. (The symbol and the reference are structur[ally] the same?) God created man to His likeness. The same thing appears at different levels. Here we have an 'unfolding'." ("Das Prinzip, daß jeder math[ematische? metaphysische?] Satz eine Verallgemeinerung für beliebig höhere Mächtigkeit hat (aber nicht umgekehrt) drückt eine der allgemeinsten Eigenschaften des Aufbaus der Welt aus. Nämlich: Alles spiegelt sich in allem. (Das Symbol und die Bedeutung sind struktur[ell] gleich?) Gott schuf den Menschen sich zum Bild. Dasselbe erscheint auf verschiedene Niveaus. Es handelt sich um eine 'Entfaltung'.") Compare *Monadology* section 83.

[93] Note that for Leibniz, what makes mathematical truths true has nothing to do with possible worlds, only with the principle of contradiction. For an argument that the notion of possibility that defines mathematical truth in Husserl's transcendental idealism satisfies the condition mentioned, see Van Atten [2001]. The particular relevance of this fact is that after 1959 Gödel adopted Husserl's transcendental idealism as a means to develop Leibniz' monadology scientifically. See the Concluding remark, below.

first domain is to suggest to us the relevant general principle governing it, so that we can apply that to the second domain. Such a principle may well hold in merely possible or fictional domains as well as in actual ones. Gödel's analogy argument may or may not presuppose that the monadology is true. That would seem to depend on whether the general principle required should involve notions specific to the monadology or not. The reason just presented why the analogy is ineffective does not turn on the answer to this question, however, for it was argued that there can be no such principle anyway. This also means that, if one makes the assumption that Leibniz' monadology (or something sufficiently close to it) is the true metaphysics, there is no direct argument either: knowing the details of exactly how sets fit into this metaphysics yields no additional means to determine the truth value of the reflection principle. Both the analogy argument and a direct argument will fail for the same reason, namely, that in Leibniz' system the specifically metaphysical principles do not imply constraints on what can be true about pure sets and collections. More generally, as we have seen, Leibniz' specifically metaphysical principles do not imply constraints on what can be true in any part of pure mathematics. The present considerations on Gödel's analogy argument and on the possibility of a direct argument are therefore not really specific to sets and reflection, and can be expected to have wider application.

In the light of the absence of implied metaphysical constraints on mathematics, it is not surprising that when Leibniz attempts to show that there can be no infinite wholes, he proceeds from logical truths and not from metaphysics or properties of minds. Contrast this to, for example, Brouwer, who based his idea that in mathematics there exist only potentially infinite constructions (and hence no constructed infinite wholes) not on a conceptual argument but on an observation about the human mind.

4.3 "Medieval ideas"

After having presented the analogy with the monadology, Gödel adds that "according to medieval ideas, properties containing V or the world would not be in the essence of any set or monad". As the reflection principle for monads follows from the monadology itself, and the analogy should then directly lead to the reflection principle for sets, this remark on medieval ideas does not seem to play a role in the argument. It seems rather an afterthought, a corraboration of the argument and its conclusion from medieval quarters.

A characteristically medieval idea (in the Christian world) is that the world and its creation are radically contingent. If the essence of any object in the world would involve the whole world, that essence might be taken to put limits on that contingency, and hence on God's freedom in creating the world. A related point is that if the essence of an object would involve the world, understood as the totality of all actual objects, it would in particular involve its own existence, but for the medievals this is only the case for God. To the extent however that one is looking for medieval ideas that could be applied to set theory, where truths are necessary and contingency plays no role whatsoever, this seems not the right suggestion for what Gödel may have had in mind.

The only idea I have been able to find that does not depend on contingency would be the idea that "being" is what medieval philosophers called a transcendental notion. This means that the notion of "being" (ens), and for example others such "one" or "true", transcend the categories into which reality can be classified because they are too general notions to define a category. The extension of the concept of being coincides with, or (if one assumes God exists but does not fall under the categories) even properly includes, the extensions of the categories combined. Aristotle already recognized the existence of such notions (*Metaphysics* 1003b25, 1061a15). The idea is therefore not medieval in the sense of having been introduced in the Middle Ages; but it is typically medieval in that the development of theories about transcendentals did not begin until then. The first systematic treatment of transcendentals is taken to be *Summa de Bono* by Philip the Chancellor, written between 1228 and 1236; but the best known passages dealing with this notion are those in Aquinas' *De Veritate* (1256–1259) and the *Summa Theologica* (1265–1272).[94]

Aquinas specifies that "the individual essence of an object is what is given by the definition [of that object]".[95] In turn, that definition consists in a specification of the genus of the object and of the specific differences that distinguishes it from other objects of the same genus. The argument that being cannot be a genus is the following: "Every genus has differences distinct from its generic essence. Now no difference can exist distinct from being; for non-being cannot be a difference".[96] The idea is that genera and differences serve to distinguish the objects that exist from one another, and hence correspond to asymmetries between them; however, no two objects that both have being can be related to being asymmetrically. Therefore, on the Aristotelian model of definitions, the concept of being cannot contribute to the definition of any object. If one understands by "the world" "all that has being", this means that the essence of no object involves the world. Indeed, Aquinas calls the multitude that results from dividing being according to all its forms the "transcendent multitude", points out that like being itself it is not a genus, and distinguishes this from "numerical multitudes" (Aquinas [1265-1274], I, 30, a.3).

Leibniz also recognizes that "being" is a transcendental notion. Usually he refers to the characteristic property of transcendentals that they are all convertible with being: that is, the transcendental terms (e.g., being, one, true) differ from one another intension but not in extension. To Des Bosses, Leibniz wrote on February 14, 1706: "I agree with you that being and one are convertible";[97] and some twenty years earlier, on April 30, 1687, to Arnauld:

> "I regard as an axiom this proposition of which the two parts

[94] From notes in his archive, it is known that Gödel read works of Aquinas.

[95] Aquinas [1265-1274], I, 29, a.2 ad 3: "essentia proprie est id quod significatur per definitionem".

[96] Aquinas [1265-1274], I, 3, a.5: "omne enim genus habet differentias quae sunt extra essentiam generis; nulla autem differentia posset inveniri, quae esset extra ens; quia non ens non potest esse differentia". See also Aristotle's argument in *Metaphysics* 998b21–28.

[97] "Ens et unum converti tecum sentio".Leibniz [1875–1890], II, p.300.

> differ only by their emphasis, namely, that what is not really *one* being is not really one *being* either. It has always been believed that one and being are reciprocal".[98]

In this last sentence, Leibniz makes an implicit reference to Aristotle and the scholastics. I do not know whether it was this reference that led Gödel to consider medieval philosophy in this context. Be that as it may, Leibniz' reason for considering "being" a transcendental was different from that of the scholastics.[99] Where the scholastics considered the notion of being as it applies to an object in the actual world, Leibniz considered the notion of being as it applies to a possible object. This notion corresponds to that of being one, as a possible object is determined by one complete concept. Leibniz' conception in terms of purely possible as opposed to actual beings (in the world) comes closer to what Gödel says when he invokes these "medieval ideas", as he wants to include sets, which for Leibniz are always possible but, being "incomplete" (i.e., never concrete), never actual objects.

The conception of being (or the world) as a transcendental, whether construed in the scholastic or in the Leibnizian sense, would indeed have the consequence that Gödel mentions, namely that no essence of a substance involves the world. But the reason why this is so would hardly be suggestive of the reflection principle. The argument from the transcendental nature of being would go through regardless of the exact properties of the universe (or of the realm of possible objects), for it depends only on an intrinsic characteristic of Aristotelian definitions. No aspect of inexhaustibility or inconceivability of the universe plays a role in it. It would seem, then, that the transcendental nature of being is compatible with both the failure and the correctness of the reflection principles for sets and for monads.

5 Concluding remark

As we have seen, Leibniz' monadology is compatible with whatever the truths of pure mathematics may turn out to be. A positive consequence of this fact is that, should a purely conceptual or internal justification for the reflection principle be found[100] this will fit into the monadology immediately. But Gödel was also interested in yet another approach. The idea here is to deepen Leibniz' monadology by considering that concepts and possibilities, though not created by God, are constituted in his mind. To Hao Wang, Gödel once complained that "some of the concepts, such as that of possibility, are not clear in the work of

[98] "Je tiens pour un axiome cette proposition identique qui n'est diversifiée que par l'accent, sçavoir que ce qui n'est pas veritablement *un* estre, n'est pas non plus veritablement un *estre*. On a tousjours crû que l'un et l'estre sont des choses reciproques".Leibniz [1875–1890], II, p.97.

[99] See also Kaehler [1979], p.119n.39.

[100] James van Aken ((1986), p.1001) observes that such an internal argument would be "a coup".

Leibniz",[101] and he stressed that "Leibniz had not worked out the theory".[102] As a means to develop Leibniz' philosophy, Gödel came to embrace and recommend Husserl's transcendental phenomenology from 1959 onward.[103] The suggestion, then, is that a phenomenological analysis of the types of acts and powers involved in the constitution of possibilities may lead to sufficient clarification of the notion of mathematical possibility to lead to a (direct) justification of the reflection principle.[104]

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[101] Wang [1996], p.310.
[102] Wang [1996], p.87.
[103] For an analysis of Gödel's turn to phenomenology, see Van Atten & Kennedy [2003]. He praised Dietrich Mahnke's *Neue Monadologie* (1917), a version of Leibniz' monadology written from a largely phenomenological point of view, as "vernünftig!", Van Atten & Kennedy [2003], p.457.
[104] As Gödel ((1961), pp.383–385) suggests for mathematical axioms in general.

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