PRESUPPOSITIONS IN MATHEMATICS

Krzysztof Wójtowicz

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THE FIRST IDEA...

- ...that perhaps comes to mind is:
- MATHEMATICAL THEORY = AXIOMS + RULES
- Presuppositions = axioms?
- Here: a more general setting
- The „basic equation” = a presupposion

PROBLEMS TO BE DISCUSSED

1. INFORMAL *versus* FORMAL
2. NO *IGNORABIMUS* IN MATHEMATICS?
3. Is mathematics *a priori?*

1. FORMAL *vs* INFORMAL

- The „semantic" tradition (Descartes): the proof = an intuitively acceptable sequence of propositions
- Both the assumptions and the steps of the proof should be **intuitively obvious**
- (Much) later: the postulate of formalizability of all proofs (Pasch, Hilbert)

IDEAL *vs* REAL MATHEMATICAL PROOFS

- IDEAL PROOFS:
- Sequences of formulas in a formalized language (e.g. ZFC or PA)
- REAL PROOFS:
- Are not formalized but convincing
- Appeal to intuition, contain gaps
- Are rational forms of mathematical argumentation

Hilbert’s bridge

- What is „Hilbert’s bridge” between the informal proof and the formal counterpart?
- Why do we assume, that proofs can/should be formalized?
- Need for clear criteria of mathematical truth
- A methodological constraint

The formalizability postulate

- A discovery concerning proofs?
- An insight into the deep nature of proofs?
  - Previously unknown
- An arbitrary stipulation concerning acceptable mathematical argument
- A (re)definition of the notion of „mathematical"
  – A new methodological criterion

The postulate...2

- PROBLEMS TO DISCUSS:
- Insight into the nature of mathematics?
- Why do we believe it is true?
- Inductive arguments?
- Logical reductionism?
- „metaphysics of proofs from the book”?

We will never know, that...

- Boolos: second-order reasonings which are not feasible in first-order logic
  - What is the status of such reasonings?
- Proofs which are too long – what is their status?
- Do we accept a proof because it can be formalized <u>in principle</u>?

Non-formal mathematics

- Why do we assume, that non-formal mathematics is OK.?
- Because it can be formalized?
- Because mathematics is **not** formalized?
- MATHEMATICIANS: maths as it is, is O.K.
- PHILOSOPHERS: maths is O.K., if formalized
- Nature of mathematics = ?

2. NO IGNORABIMUS

- Hilbert: *„Wir müssen wissen. Wir werden wissen."*

1. Mathematical problems are solvable
2. Mathematics is consistent
3. Mathematics is objective

Gödel

- Gödel: well-formulated mathematical problems are solvable
  - Gödel's theorems are not a problem
- Objective $\neq$ subjective mathematics
- Our intuition develops and leads to new insights
- We can solve formally unsolvable problems by passing to stronger theories

Continuum = ?

- CH – independent from ZFC (Gödel, Cohen)
- Gödel's program for new axioms
- Solvability in the broad sense: not within a formal framework, but by laying out a new formal framework
- Gödel's square axioms
- Woodin's program

No *ignorabimus*

- Mathematical statements are objective
- Realism in truth-value
- Mathematics is consistent
  - The inconsistencies are only local in character

3. IS MATHEMATICS A PRIORI...?

- THE RECIEVED VIEW:
- a priori
- independent of sensory experience
- proving theorems = a purely rational activity
- grasping inferential connections
- *intelektuelle Anschauung*

EMPIRICAL ELEMENTS?

- Can a mathematical proof contain an empirical ingredient?
- Can there be statements about numbers, which have empirical justification (without being provable)?

COMPUTER-ASSISTED PROOFS (CAPs)

- 4-color theorem (4CT)
- Colouring of maps: no adjacent countries have the same color
- Formulation: 1852
- Solution: 1976 (Appel, Haken, Koch)

1. Calculations performed on a computer
2. 1200 hours

QUESTIONS CONCERNING CAPs

- 4CT: were the inferential connections between premises and conclusions really proved?
- Is the algorithm (proof) logically correct?
  - Has a classical counterpart
- Are the laws of electrical engineering reliable?
  - Is the electronic device reliable?
  - 4CT relies on a physical experiment
  - Have no classical counterpart

A PRIORI?

EMPIRICAL ELEMENTS IN PROOFS

- Are CAP’s problematic?
- Problem of reliability
- New kind of mathematical argumentation?
- „Quantum proofs”
- Hypercomputational procedures
- Would these (hypothetical) procedures be accepted as mathematical?

PROBLEMS TO DISCUSS

1. INFORMAL / FORMAL discourse in mathematics
2. NO *IGNORABIMUS* in mathematics?
3. EMPIRICAL ELEMENTS *versus* the A PRIORI VIEW about mathematics?

THE VERY LAST SLIDE

- THANK YOU FOR YOUR ATTENTION