

Zbigniew Król

## SCIENTIFIC HERITAGE: THE RECEPTION AND TRANSMISSION OF EUCLIDIAN GEOMETRY IN WESTERN CIVILIZATION<sup>1</sup>

### ABSTRACT

This paper presents sources pertinent to the transmission of Euclid's *Elements* in Western medieval civilization. Some important observations follow from the pure description of the sources concerning the development of mathematics, e.g., the text of the *Elements* was supplemented with new axioms, proofs and theorems as if an “*a priori* skeleton” lost in Dark Ages was reconstructed and rediscovered during the late Middle Ages. Such historical facts indicate the apriority of mathematics.

**Keywords:** Euclid's elements; apriority of mathematics; Western Civilization.

The geometry in Euclid's *Elements* is not only a part of pure science. The *Elements* form also a very important part of Western culture (here, equivalently, Western civilization). However, between antiquity and the first printed edition of the Latin text (1482) and of the Greek text in 1533 (cf. *Elementa geometriæ*, Basle: Johann Herwagen, 1533 A.D.), there was a manifold of traditions concerning manuscripts, translations and editions of the text of the *Elements*. The analysis of these sources unveils in detail also ways and mechanisms of the creation of mathematical knowledge. The *Elements* were not an *a priori*, ahistorical construction by the human mind. The historical *Elements* contain traces of many thought experiments, ways of thinking and attitudes towards mathematics in different times. Therefore, it is important to know something about the history of the text (or rather of *a text*) which influenced so much the European culture.

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From the pure description of the sources there follow some important observations concerning the development of mathematics. For instance, as we will see, in European science there were no proofs known in geometry up to the XII century A.D. (except from the three first theorems from the first book of the *Elements*). Certain Latin translations from Arabic, i.e., not those from Greek, were the most important and influential in Western civilization. The text of the *Elements* was supplemented with new axioms, proofs and theorems, i.e. the “*a priori* skeleton” was reconstructed and rediscovered. A more detailed inquiry indicates that *intuitive foundations* of the Euclidean geometry were changed in the process of transmission.

Proclus, whose writings are important sources for the modern history of mathematics, did not influence medieval mathematics. Proclus’ *Commentary on the first book of Euclid’s ‘Elements’* was edited together with the Greek text of the *editio princeps* in 1533. However, there are some other ancient authors, especially Heron or Simplicius who were more important in the medieval mathematics and philosophy of mathematics. The discussion concerning their views created historically an essential part of the transmission of Euclidean geometry and the *Elements* of Euclid in medieval Europe.

The main source for ancient commentaries of the *Elements* and certain views of some ancient authors, namely Heron, Simplicius, Boethius, Agapius is the medieval Arabic *Commentary of al-Nayrizi of Euclid’s Elements of geometry*. We know only two survived Arabic manuscripts containing *Commentary*, i.e. the *Codex Leidensis* MS OR 399.1 (Ms L) and the manuscript *Qom* 6526, (Ms Q). The first codex is described by G. Junge, J. Raeder and W. Thomson in [Anzulewicz 1999], pp. 206–210; cf. also [LoBello 2003a], pp. 82–85. It was edited by R. O. Besthorn, cf. [Besthorn 1932]. The second codex, discovered only recently 1992 (cf. [Brentjes 1992]), remains unedited, except the Arabic text of *Book I* of the *Commentary* by Rüdiger Arnzen (cf. [Arnzen 2002]). Some information about it is in [Arnzen 2002] and [LoBello 2009]. Arnzen based his edition on the two above mentioned Mss. Both manuscripts have some lacunas and, for instance, the *Codex Leidensis* breaks off at the first part of Book VII and it is incomplete in the part concerning the definitions of the Book I of the *Elements*. Fortunately, the *Qom* MS contains almost the whole part of the commentary with the definitions from the first book of the *Elements* which is missing from the *Codex Leidensis*, except the definitions I.1.–I.3. Also, partially the same material as the Arabic al-Nayrizi’s *Commentary* (i.e. the Mss L and Q) contains the manuscript from Patna, Patna HL 2034 from the Khuda Bakhsh Oriental Library in Bankipore with the commentary on Euclid’s *Elements* by Ahmad bin Omar al-Karabisi (X century A.D.); more information see [Arnzen 2002], pp. XVII–XVIII. There is also one more and unedited Arabic manuscript with the al-Karabisi’s *Commentary* in Rasht in Iran. Al-Karabisi’s *Commentary* is limited to the introductory part of the first book of the *Elements*, mainly to the definitions and general introductory material; cf. [Arnzen 2002], p. XVIII.

It is not possible to reconstruct the whole text of the *Commentary* from the only Arabic texts of the both aforementioned manuscripts. Nevertheless, we have also some Latin sources containing a translation of the *Commentary*. The famous translation of Gerard of Cremona (1114–1187) is the most important one. This Latin commentary has, for instance, the text (and some comments) of the definitions Def. I.1–I.3 which is missing from the Arabic sources. The Arabic texts end at the beginning of the book VII and the Latin text of the commentary preserves all ten books of it. In general, the text of the Arabic version of the *Elements* is not translated by Gerard into Latin. He translated (almost) only the commentary.

The four manuscripts of Gerard's *Commentary* are known: Biblioteka Jagiellońska 569, Cracow, f. 1–23 (pp. 7–51), (XIV century; Ms **K**), Biblioteca Nacional 10010, Madrid, f. 13v–36v–, 49v–50, (XIII/XIV century; Ms **M**), Bodleian Library Digby 168, Oxford, f. 124–125, (XIV century (abridged version)), and Regin. lat. 1268, Vaticano, f. 144–183v, 206r–207v, (XIV century; Ms **V**). The *Manuscript Cracoviensis* was edited by M. Curtze and J. Heiberg in Leipzig in 1899; cf. [Heiberg 1839], vol. VIII (M. Curtze, *Anarithi in decem libros priores Elementorum Euclidis commentarii*, pp. 1–252).<sup>2</sup> Tummers has shown that MS **V** is copied from **M** and that **K**, **M** and **V** are based on the other (unknown) common source; cf. [LoBello 2003b] p. xxx and Tummers, *op. cit.* S. Brentjes reports that some extracts from the commentary are found also in the manuscript in Mumbai (Mulla Firuz Collection in Mumbai, R I.6, dated by Brentjes on the Xth century, cf. [Brentjes 2001]). The same manuscript preserves also more than some short fragments from the al-Hajjaj II tradition (see below).

The Qom Manuscript is dated to the XV century. It is necessary to add that the Qom MS has mainly free space left for diagrams and only a few of them are inserted into the manuscript (cf. [LoBello 2009], p. xiii). There is no one and a new diagram in the part of the book I which is present in Ms **Q** and absent in Ms **L**.

The comparison of the Arabic with the Latin version of the commentary leads to a conclusion that they both preserve the same text of the *Elements* which is own al-Nayrizi edition supplemented by comments; cf. [Brentjes 2001] and [LoBello 2003b], p. xxxii. Brentjes argues, however, that the direct sources of the Latin and Arabic manuscripts are different. No Latin manuscript is an original version prepared by Gerard; they all are a result of some later editorial activity. As it concerns ancient authors interested us at the moment, the Latin Gerard's commentary transmits more from Heron's comments than the Arabic sources.

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<sup>2</sup> Cf. also the edition of the book I–IV by P. M. J. E. Tummers in [Tummers 1984] and [Tummers 1994].

The last part (cf. op. cit., pp. 252–386) of Curtze's edition of the text of Ms **K** of al-Nayrizi's on commentary contains one more commentary on the book X of the *Elements*, i.e. Muhammad ibn 'Abd al-Baqi. This commentary was used in the text for the book X in the manuscript Vat. Reg. lat. 1268; cf. [Busard 1985], p. 135.

Many parts of the translation of *Commentary of al-Nayrizi of Euclid's Elements of geometry* were copied and used by Albertus Magnus in his *Commentary on book I of Euclid's Elements of geometry*; cf. the critical edition in [Tummers 1984] and an English translation in [LoBello 2003c]. The main codex with this commentary is the *Codex Vienna* Dom. 80/45. Moreover, the commentary was the source for Roger Bacon and Campanus of Novara. The latter is more important for us. H. L. L. Busard indicated such places (and manuscripts) explicitly; cf. [Busard 1985], pp. 134–135.

Obviously, the sources indicated above are based on some others manuscripts and traditions. Moreover, a commentary usually contains a commented variant of the text of the *Elements*. Summing up the findings of Tummers and Arnzen, one can reconstruct the lines of the development of al-Nayrizi's tradition. Heron used the original, pre-Theonian Greek text of Euclid's *Elements* and the text of the *Elements* used by Heron, and this material is partially present in al-Nayrizi *Commentary*. Al-Nayrizi uses a pre-Theonite text of the *Elements* which also was used by the Greek commentators on Euclid: Heron, Simplicius, Pappus (and—one can add—also Proclus who is, however, not present in the al-Nayrizi's commentary). Moreover, al-Nayrizi uses also the Theonite tradition through a Greek post-Theonian text of the *Elements* which was the source for the Arabic al-Hajjaj translation. The same Theonian Greek text was the base of the Ishaq-Thabit Arabic tradition (see below) which is itself the source for the Latin translation by Gerard of Cremona. Ms **L** and Ms **Q** had a common ancestor. Also, both lines of the transmission of al-Nayrizi's *Commentary*, i.e. the Arabic and the Latin, had a common ancestor which is also a common ancestor for their more direct ancestors; cf. Ms  $\phi$  in [Arnzen 2002], p. XXV.

It is necessary to explain now also general lines of the transmission of the text of the *Elements* from antiquity to the Middle Ages. This survey will provide a chronology for the emergence of a new intuitive model of geometry.

### GREEK EUCLID IN MEDIEVAL EUROPE

It is a well-known fact that almost all known manuscripts of the *Elements* (which were the main source for the later editions of the *Elements* up to the XIX century) belong to the tradition stemming from the edition by Theon of Alexandria dated about 364 A.D. Theon inserted some supplementing material, corrected some theorems and proofs. Theon's improvements and changes are briefly described in [Busard 1987], pp. 5–7. For instance, we know from Theon's commentary on Ptolemy's *Almagest* that, and also how, Ptolemy cor-

rected the theorem VI.33 of the original Euclid's *Elements*. This explicit information made it possible to find the unique manuscript with the pre-Theonite Greek text of the *Elements*: Ms. Vat. Gr. 190 from the IX century (designated as **P** in Heiberg's critical edition of that manuscript; cf. [Heiberg 1889]) by F. Peyrard in 1808. All others manuscripts belong to the Theonian tradition. In general, there are two lines of the transmission of the *Elements* to Latin Europe. The first contains the text of the *Elements* translated directly from the Greek or at least originating in such a translation. The second is based on some Arabic sources. At first, I am going to describe briefly the first of the lines: that of the editions made directly from the Greek *Elements*.

To the first line based on a Greek (Theonian) text of the *Elements* belongs the Boethian tradition which is very complex in itself. The main lines of the development of this tradition are reconstructed by M. Folkerts; cf. [Folkerts 1970]. It is very difficult to decide if Boethius translated all 13 (or even 15) books of the *Elements*. However, the material indicates that his translation contained at least the definitions, postulates, axioms from books I–V, many statements of the propositions from books I–IV but without any proofs. The only known proofs are the proofs of three propositions: I.1–I.3. Some of the material of the above translation was copied into the *Codex Agrimensorum* around 550 A.D. by the Benedictine monks at Corbie and St. Omer. There are known 10 manuscripts of the *Codex Agrimensorum* (cf. [Folkerts 1989] and [Folkerts 1970], Chapter II.C *Boethius' Geometrie II und die Schriften der Agrimensoren*:

1. Bürgerbibliothek 87 in Bern, **be** in Folkerts, [Folkerts 1970], written in the year 1004 A.D., f. 16v, (contains only the propositions I.1–I.3 with proofs),
2. Bibliothèque Royale 4499–4503 in Bruxelles, **b** in Folkerts, from the XII century (in.), f. 4v–5, (contains only the definitions I.19–I.23),
3. Bibliothèque Royale 10629–10660 in Bruxelles from the XII century, (copy of Vaticano, Palat. lat. 1564; see below),
4. Bibliothèque de la Ville 498 in Chartres, **c** in Folkerts, from the XII century, f. 153, s.XII, (contains only the definitions I.19–I.23),
5. Bayerische Staatsbibliothek CLM 13084 in München, **y** in Folkerts, from the X century A.D., f. 53v–54, (contains only the definitions I.19–I.23, postulates 1–5 and axioms I.1, I.3, I.2 and I.7),
6. Biblioteca Nazionale in Napoli V A 13, **N** in Folkerts, from the X century (in.) A.D., f. 30v–31, (contains only the propositions I.1–I.3 and their proofs),
7. Bibliothèque Nationale in Paris Lat. 8679A from the XVI century, f. 19–21, (this is a copy of the Wolfenbüttel Herzog A. Bibliothek Cod. Guelf. 105 Gud. lat., see below),
8. Vaticano Ottob. lat. 1862 from the XIIth century A.D., **n<sub>3</sub>** in [Folkerts 1970], f. 26, (contains only the definitions I.19–I.23); the codex is designated as “**P**” by Thulin (cf. [Thulin 1911]),
9. Vaticano Palat. lat. 1564 from the IX century A.D., f. 79–82; the manuscript is designated as **n<sub>3</sub>** by Folkerts in [Folkerts 1970],

10. Herzog August Bibliothek in Wolfenbüttel, Cod. Guelf. 105 Gud. lat., G in Thulin and Folkerts, from the IX century A.D., f. 43–45v.

For the exact description of the main codices and manuscripts, see [Thulin 1911] and [Folkerts 1970]; for the critical edition of the text of *Codex Agrimen-sorum Romanorum*, see [Thulin 1913].

The extant part of the Boethius translation in the above codices is: definitions I.1–12, I.14, I.13, I.15–23, the postulates I. 1–5, axioms I.1, I.3, I.2, I.7, and the propositions I.1–3 with proofs; cf. [Folkerts 1989].

On the other hand, one can find the fragments of Boethius' translation of *Elements* in the 6 survived manuscripts of (the "third" recension of) the *Institutiones* of Casiodorus, i.e., the definitions of *Book I* (these are the definitions: I.1–19, I.14, I.13, and I. 15–23, the order of the definitions is as in the text), V (the definitions: V.1–8, V.11, V.9, V.10, V.13, V.12, V.14–16, V.18–17) and the definition II.2, the postulates (1–5) of book I, and the axioms (I.1, I.3, I.2, I.7) from *Book I* and *V*. This edition was prepared at Corbie in the VIII century A.D.

Menso Folkerts writes:

“Die 3 cassiodor-Rezension ist wahrscheinlich im 8. Jahrhundert, auf jedem Fall vor Hrabamus Maurus entstanden. Auch ‘Boethius’ Geometrie I stammt vermutlich aus dem 8. Jahrhundert, während die ältesten Mb-Codices im 9. Jahrhundert geschrieben wurden. Der Archetypus Ma, Mb, Mc muss also spätestens im 8. Jahrhundert entstanden sein. Möglicherweise geht er auf Boethius’ Euklidübersetzung zurück” (See [Folkerts 1970], p. 72).

The manuscripts of the *Institutiones* are listed by Folkerts in [Folkerts 1989], cf. also [Mynors 1961] and [Folkerts 1970]:

1. Cheltenham (olim), Phillipps MS 16278 from the IX century A.D.,
2. Hunterian Museum in Glasgow 281 from the X/X century A.D.,
3. Badische Landesbibliothek in Karlsruhe Augiensis 106 from the X century A.D.,
4. Bayerische Staatsbibliothek in München CLM 13084 from the X century A.D., f. 54–54v, **m** in [Folkerts 1970], (extant are only the definition II.2, and V.1–8, V.11, V.9, V.10, V.13, V.12, V.14–16, V.18, V.17),
5. Bibliothèque Nationale in Paris Lat. 12963 from the X century,
6. Bibliothèque Municipale in Valenciennes 195 from the IX century.

At the same place and time, i.e. at the Benedictine monastery at Corbie in the VIII century (see [Folkerts 1970], pp. 73, 81.), a monk prepared the excerpt of *Ars Arithmeticae et Geometrie* from which the so-called *Geometrie I* version of the Boethius' *Elements* is reconstructed. We have 27 relevant manuscripts directly linked with the *Geometry I*. The manuscripts are described by M. Folkerts

in [Folkerts 1989]. One manuscript is in the University Library in Wrocław, Poland (Rehdig. 55 from the XV century, f. 2–26). Folkerts summarizes that the extant Boethian part contains the definitions I.1–12, I.14, I.13, I.15–23, II.1, II.2, III.1–6, III.8–11, IV.1, IV.2, the postulates (I.1–5), axioms (I.1, I.3, I.2, I.7) and the propositions: I.1, I.2–4, I.6–8(9), I.10–18, I.21, I.23, I.26–28, I.31–37, I.39–41, I.43, I.42, I.44–48, II.1, II.3–6, II.9–12, II.14, III.3, III.7 (beginning), III.22 (end), III.27, III.30–33, IV.1–4, IV.6, IV.8, IV.12, IV.11, and again III.7 (end), III.9, III.12, III.10, III.13, III.14, III.16, III.18, III.19, III.24, III.22 (beginning). All the above propositions are without proofs.

There are also two more sources which probably belong to the Boethian tradition, cf. [Folkerts 1989]. The first is a very old manuscript (a palimpsest) from Verona (Biblioteca Capitolare ca. A.D. 500 XL (38), f. 331v–331, 326v–326, 341–341v, 338–338v, 336–336v, 343–343v) edited by M. Geymonat; cf. [Geymonat 1964], in which the following propositions are extant: XI.24–25, XII.2–3, XII.8, XIII.2–3, and XIII.7; cf. also [Folkerts 1989]. The second, i.e. the Munich manuscript from the IX century (Universitätsbibliothek 2, 757, f. 1–2v) contains theorems I.37, I.38 and II.8, II.9 (beginning); cf. [Geymonat 1967].

The *Geometry II* from the XI century was compiled in Lorraine. The monk who created the *Geometry II* used the same manuscript as the compiler of *Geometry I*, however, he added the proofs of the theorems I.1–I.3 and copied also a Boethian version of enunciations of the 10 theorems from the book I which are missing from the *Geometry I*, i.e. the theorems I.5, I.7, I.19, I.20, I.22, I.24, I.25, I.29, I.30, I.38.

There are known 23 manuscripts of this version; cf. [Folkerts 1989].

The Boethian version of the *Elements* was a predominant version based on a direct translation from a Greek text into Latin in the Middle Ages up to early XII century; cf. [LoBello 2003c], p. 12. Some parts of the Boethian tradition are present in the Adelard II and III (Latin) versions of the *Elements*; see below.

One more translation made directly from the Greek came out in the XII century. We know 3 manuscripts containing this version (cf. [Folkerts 1989], [Folkerts 1970]):

1. Bibliothèque Nationale in Paris, Lat. 7373 from the XIII century, f. 2–175v,
2. Biblioteca Nazionale Centrale in Firenze, Conv. soppr. C I 448 from the XIV century, f. 1–104v, (the version ends within the theorem X.48),
3. Biblioteca Nazionale Marciana in Venice, *fondo antico* 271 (= 1642) from the XV century, f. 1–1v, (contains only the definitions I.1–I.20).

The translation was prepared after the year 1160 in the Norman kingdom in Sicily and this translation is dependent, in some places, on the codex from Oxford, Bodleian Library D’Orville 70 (XIII/XIV century, (i.e. “B” in Heiberg’s edition of the *Elements*) which belongs to the Arabic-Latin tradition of Adelard of Bath; see below. According to Busard, the Greek source of this version was

contaminated with Theonine and ante-Theonine traditions; cf. [Busard 1987], p. 10. Heiberg conjectured that there must have been such intermediate manuscripts in the XI century. One of them, i.e. Heiberg's "S" manuscript of the Escurial Library (III-5), had to be such a manuscript; cf. [Busard 1987], p.10.

This version of the *Elements*, which is a complete version of Euclid's text translated directly from the Greek, was not influential. A critical edition of it, see [Busard 1987]; cf. also [Murdoch 1967].

### EUCLID IN ARABIA: ARABIC SOURCES FOR MEDIEVAL MATHEMATICS

Translations belonging to the Arabic-Latin tradition were more important and popular. They create the second line of the transmission of Euclid's *Elements*. So, first of all, it is necessary to say something about the Arabic sources.

The first Arabic translation of the *Elements* made from a pre-Theonian source coming from the Byzantine Empire is that of al-Hajjaj bin Yusuf bin Matar prepared during the reign of the caliph Harun al-Rashid, i.e., between the years 786–809. There are no known manuscripts with this al-Hajjaj I version; the version seems to be lost; cf. [Folkerts 1989]. However, al-Hajjaj translated the *Elements* twice, according to the list of the *Fihrist*. It was unclear if the second al-Hajjaj version was preserved in the *Commentary* of al-Nayrizi. M. Folkerts suggests that it is extant in the aforementioned *Codex Leidensis*; cf. also [LoBello 2003a], pp. 20–21, 25, 30–31. Therefore, it is possible to know the books I–VII (beginning) of this version of *Elements*. Folkerts writes: “*There are manuscripts in Leningrad, Teheran and in the El Escorial which probably also carry a text similar to Kobenhavn 81 [i.e. Kobenhavn, Kongelige Biblioteket 81 from the XI–XIII centuries—Z.K.], but they have not yet been investigated [i.e. in the year 1989–Z.K.]*.”<sup>3</sup> Folkerts also adds (*ibidem*): “*Some of the Ishaq-Thabit manuscripts (...) seem to be contaminated with a Hajjaj text, especially mss El Escorial, ar. 907; Leningrad, C 2145; Teheran, Majlis Shura 200.*”

Moreover, the text of Euclid in the *Codex Leidensis* is not an abbreviated version of the *Elements* as it should be according to the sources. Moreover, the recent investigation of the manuscript leads to the view that this text is a compiled mixture of al-Hajjaj and (the later) Ishaq-Tabit texts “decorated” adorned with some commentaries taken from the Arabic translations of some Greek commentaries, Arabic and (mainly) al-Nayrizi's commentaries; cf. [LoBello 2003b], p. xv. The current state of affairs as to the problem whether al-Nayrizi's commentary contains al-Hajjaj II text of the *Elements*, is given above, see recent remarks concerning the commentary of al-Nayrizi.

However, the problem remains: if and where can one find the extant parts of al-Hajjaj II commentary in the Arabic sources? Currently, some parts of the text

<sup>3</sup> Cf. [Folkerts 1989].

of al-Hajjaj II are identified in the following manuscripts (cf. [Brentjes 2001], [LoBello 2003b]):

1. St. Petersburg Akademija Nauk, Institut Vostokovedenija C 2145 (dated on the year 1188 A.D.). The Ms contains a part of the book VIII of the *Elements* of the al-Hajjaj version.

2. Istanbul Süleymaniye Kütüphanesi Fatih Camii 3439.1 (dated: 1190–1191 A.D.). The Ms contains the books XI–XIII of the version.

3. Chester Beatty Library in Dublin, Arabic 3035 (dated: 1270–1271 A.D.), ff. 126, it contains the text of the books I.39–XV (mainly of Ishaq-Thabit), but only a small part is of al-Hajjaj's readings.

4. Kongelige Biblioteket in Kobenhavn, Ms 81, it contains the text of the books V–XV, however, only the books XI–XIII are contaminated by the al-Hajjaj II version mixed with Ishaq-Thabit's text.

5. Bibliothèque Nationale in Paris, MS Persan 169 (dated: 1500–1625 A.D.). The manuscript contains the pure text of the al-Hajjaj II version of the book II. S. Brentjes (cf. [Brentjes 1994]) who investigated the manuscript indicates the strong connection between this manuscript and the Andalusian family of manuscripts (which belongs to the next, Ishaq-Thabit, edition of the *Elements*, see below).

6. Mulla Firuz Collection in Mumbai, R I.6, dated by Brentjes on the X century, cf. [Brentjes 2001]. The manuscript preserves the text of the *Elements* and “it is the first Arabic manuscript from the Muslim East which contains more than a brief extract from a version affiliated with the Hajjaj tradition ...” (op. cit., p. 46). (The manuscript contains also the extracts from the text of the *Elements* according to the al-Nayrizi's commentary.)

7. Majlis Shura 200 from Tehran (226ff.), contains the books I–XV. The manuscript is close to the Chester Beatty Arabic 3035 (above) and the next manuscript.

8. Arshi 200 from Rampur, Raza Library (214ff., dated to the end of X century), contains all fifteen books of the *Elements* of the Ishaq-Thabit recension mainly.

9. Thurston 11, Bodleian Library in Oxford (212ff., 1257 A.D. (or A.H. 635)), contains the books I–XV; the books XI–XIII are of the Hajjaj version.

10. El-Escorial Derenbourg ar. 907 in Madrid, (dated the XIII century), text mainly of the Ishaq-Thabit tradition, however, it contains some short quotations from the Hajjaj version (mainly alternative proofs for the theorems VIII.20–21, alternate enunciations of the theorems II.1–9, and some additional cases for the proofs of the theorems III.24, 32, 34–36.

Some more parts of the al-Hajjaj II tradition are used in the Latin tradition of Adelard I, see below. Folkerts writes (in [Folkerts 1989]):

*“Which Arabic translation or mixed version was used in the translation of Adelard I can only be decided after a detailed comparison of Latin against Arabic. Busard (cf. [Busard 1983], pp. 18–19 – Z.K.) has shown that there is an almost literal agreement between Adelard I and the fragment of a Syriac redaction of book I which has come down to us. Two theorems compared in detail by R. Lorch (cf. [Lorch 1987]) reveal, with one uncomfortable exception, affinities with the Hajjaj phraseology. But further investigations by Kunitzsch (cf. [Kunitzsch 1985], p. 119) give quite contrary indications, i.e., that Adelard I depends on Ishaq-Thabit rather than Hajjaj. Kunitzsch has noted that Adelard I has some sections of a complete literalness against the Arabic, while other sections show a high degree of ‘‘literary Latin’’ transformation. Clearly, the text has been reworked.”*

The second author who translated the *Elements* into Arabic was the Syrian scholar Ishaq bin Husain who died in 910–911 A.D. He used a text following Theon’s edition. His translation, of which no Arabic manuscript survived, was more literal than the previous two al-Hajjaj versions. During the lifetime of Ishaq, a mathematician Thabit bin Qurra made a reedition of the Ishaq text. Actually, we know 19 manuscripts with this version contaminated by the other traditions, e.g. al-Hajjaj tradition (see above); cf. [Folkerts 1989].

The oldest manuscript with the Ishaq-Thabit recension is Malik 3586 from Tehran (240ff., dated 954/55 A.D.). The manuscript contained books I–XV. The missing part of book VII is preserved in the second part of the same manuscript, i.e. Danishgah 2120 also in Tehran (6 ff.); cf. [Folkerts 1989]. Folkerts (op. cit.) and Lo Bello, cf. [LoBello 2003b], pp. xiii–xxix) lists the other manuscripts which can be divided into two another sub-lines. The first of them contains mainly parts of the pure Ishaq-Thabit text. To this first group, besides the two manuscripts mentioned already, belongs also, for instance, the manuscripts (Folkerts, op. cit., gives a more complete list):

1. Huntington 435, Bodleian Library in Oxford, (217ff., dated on the XII century), contains the text of the books I.13–XV,
2. O. Vet. 20 from Universitetsbibliotek in Uppsala, (202ff., X century)) which contains the text of the books I–XV,
3. al-Malik 1101 from Rabat, (ca. 200ff., dated 1284–1285 A.D. (A.H. 683), contains the books I–XV,
4. Addit. 1075 from the Cambridge University Library, (233ff., XIII century), the text starts at the definition I.19 and contains fifteen books,
5. al-Malik 5317 in Rabat, (63ff., dated A.H. 1016), contains the books I–VIII, (according to S. Brentjes, the manuscript has to be based on a similar source—containing the Hajjaj II version of the *Elements*—as the source for the so-called Andalusian family of manuscripts, i.e., together with al-Malik 1101 and El-Escorial Derenbourg ar. 907).

The second sub-line contains the Ishaq-Thabit version mixed with the fragments of al-Hajjaj text. To this group belongs the Arabic manuscripts listed already, those of the al-Hajjaj tradition above. M. Folkerts writes:

*“In the thirteenth century the two texts—al-Hajjaj and Ishaq b. Hunayn improved by Thabit ibn Qurra—were available to Nasir al-Din al-Tusi. He made several remarks about the differences between them in his Tahrir, of which we have several manuscripts. There is another tahrir, also attributed to Nasir al-Din and printed in Rome in 1594, which gives us other information about the two translations. The thirteenth century redactor tells us that certain theorems were omitted by Hajjaj but are included by Ishaq-Thabit. [Note 31: See C. Thaer, Die Euklid-Überlieferung durch al-Tusi, in: Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Abt. B: Studien, III (1936) 116–121. The propositions are: I.45; VI.12; VII.24, 25 (in Thabit’s numbering; not present in Heiberg’s Greek); X.27, 28 (in Heiberg; X.21, 22 in Thabit).]”* (cf. [Folkerts 1989]).

### THE ARABIC-LATIN TRADITION

Now we can describe the main four lines of the transmission and reception of the Latin text of Euclid’s *Elements* based on the above Arabic traditions. Firstly, there were three Latin translations from the Arabic in XII century that of Adelard of Bath, Hermann of Carinthia and Gerard of Cremona.

According to Busard, about twenty years after the first Latin translation of Adelard of Bath, in the late thirties of the first half of the thirteenth century, Hermann of Carinthia made a translation of the first twelve books of the *Elements* from the Arabic source containing an al-Hajjaj version. This edition was not very influential and it depends in some parts (e.g., in some definitions from the book I) on the next, Adelard of Bath’s, translation. Busard published a critical edition of the Hermann’s text; cf. [Busard 1968] and [Busard 1977] which survived in the three relevant manuscripts, i.e.:

1. Lat. 16646 in Bibliothèque Nationale, Paris, (f. 2–108, from the XIII century), contains books I–XII,
2. Digby 174 in Bodleian Library, Oxford, (f. 160–160v, from the XIII century), a short fragment of the theorems XI.1–4 is in Hermann’s version,
3. Regin. lat. 1268 in Vaticano, (f. 113r–113v, from XIV (in.) century), as above, it contains the theorems XI.1–4, in agreement with the first and basic manuscript, i.e. Digby 174. (cf. [Folkerts 1989]).

The next translation, Gerard of Cremona’s, is known from seven manuscripts. Busard prepared a critical edition of the text, cf. [Busard 1984]. He writes about Gerard of Cremona’s translation:

*“Although none of the seven extant manuscripts, four of which contain all of the ‘Elements’, bears the name of the translator, A. A. Björnbo ascribed this translation to Gerard of Cremona on the ground of the various linguistic criteria, which he drew from works acknowledged as Gerard’s. It is unfortunate that in the Middle Ages the Gerard translation of the ‘Elements’ had been less used and was less influential than the more inaccurate Adelard II version, for it furnishes a text which is superior to the other Latin translations and which is the closest to the Greek tradition of all Arabic-Latin versions”* (Cf. [Busard 1984], p. XII).

Folkerts adds:

*“A comparison of selected passages of the Gerard text with Arabic manuscripts [Note 35: By R. Lorch; see ... [R. Lorch, ‘Some Remarks on the Arabic-Latin Euclid, in: Adelard of Bath. An English Scientist and Arabist of the Early Twelfth Century’, ed. by Charles Burnett (London: The Warburg Institute, 1987)—Z.K.] show that Gerard basically follows the Ishaq-Thabit version. The source manuscript for his translation must, however, have included borrowings from the Hajjaj version because, inserted in the Ishaq-Thabit text translated by Gerard, we find isolated words or formulas or even passages of several words which are recognizably in the Hajjaj wording. There are some alternative proofs ‘from another book’, Gerard says, which may have been taken from Hajjaj”* (Cf. [Folkerts 1989]).

Folkerts lists the following manuscripts:

1. Bibliothèque de la Ville 196 in Boulogne-sur-Mer, from XIV century, **M** in [Busard 1984], the pages 1r-144r contain 15 books of the *Elements*, the pages 144r-147v contain scholia I-XIX,
2. Stadsbibliotheek 521 in Bruges, from the XIV century, **B** in [Busard 1984], the pages 1r-109v contain Books I-XV, 109v-113v contain scholia I-XVII,
3. Biblioteca Nazionale Centrale Conv. soppr. J V 30 in Florence, from XIV century, not used in [Busard 1984], pp. 55-57v contain scholia IV-XVII, and XX-XXII,
4. Biblioteca de la Universidad 102 (117-z-6) in Madrid, from XIV century, not used in [Busard 1984], the pages 137-196v contain Books I-X.101,
5. Biblioteca Ambrosiana D 186 inf. in Milano from the XV century, not used in [Busard 1984], the pages 1-107v contain Books I-X.109, and XIII.11-XV. 5, the pages 107-112 contain scholia I-XVII,
6. Bodleian Library Digby 174 in Oxford, from the XIII century, **D** in [Busard 1984], the pages 160v-173v contain Books XI.5-XIV.1,
7. Bodleian Library Canon. misc. 145 in Oxford, from the XV century, not used in [Busard 1984], the page. 307 contains the fragments of Book III: Definitions, Porism III.1, and the enunciations of Theorems III.2-III.7,

8. Biblioteca Comunale 2 Qq E 98 in Palermo, from the XIV century, not used in [Busard 1984], the pages 1–83v contain the Books I–X.17,

9. Bibliothèque Nationale Lat. 7216 in Paris, from the XV century, **P** in [Busard 1984], the pages 1–107v contain 15 books of the *Elements*, the pages 107v–108 contain scholia XVIII–XIX,

10. Bibliothèque Nationale Lat. 7377B in Paris, from the XIV/XV centuries, not used in [Busard 1984], the pages 1–9 contain scholia I–XVII, twice the scholium XVII: “at the end and after scholium III” (cf. [Folkerts 1989]),

11. Bibliothèque Nationale Lat. 11247 in Paris, from the XVI century, not used in [Busard 1984], the pages 78–91v contain scholia IV–XVII, XX–XXII,

12. Biblioteca Vallicelliana F. 86 in Rome, from XIV century, not used in [Busard 1984], the pages 49–54 contain Books I–I.5),

13. Lat. 7299 in Vatican from the XIV century, **L** in [Busard 1984], the pages 1r–136 contain 15 books of the *Elements*, and the pages 136–141 contain scholia I–XVII,

14. Regin. lat. 1268 in Vatican, from the XIV century, **V** in [Busard 1984], the pages 113v–142v contain Books XI.5–XV), the pages 142v–143 have scholia I–III,

15. Regin. lat. 1904 in Vatican, from the XV/XVI century, not used in [Busard 1984], the pages 86–90 contain Books XV.1–XV.5, XIII.16,

16. Rossiano 579 in Vatican, **R** in [Busard 1984], from the XIV century, the pages 1–126v contain 15 books of the *Elements*, and the pages 126v–130v have scholia I–XVII.

The most complex chronologically is the first Arabic-Latin tradition which is commonly ascribed to Adelard of Bath. M. Claget gathered the diversity of the texts containing this traditions in three groups: Adelard I, Adelard II and Adelard III; cf. [Clagett 1953]. However, it soon appeared that the picture consisting only of the three versions above is too simple because J. E. Murdoch showed that there are some other sub-lines in them; cf. [Murdoch 1968]. Moreover, Adelard of Bath is an author of only the first version.

Adelard I uses the Hajjaj version of the *Elements* and it is very close to the text of the commentary of al-Nayrizi. M. Folkerts writes about the Adelard I version: “*From this text we have four manuscripts which contain books I–VIII, one for books X.36–XV.2, one for book X.36–49, but none at all for books IX to X.35.*” (Cf. [Folkerts 1989].) He adds also:

*“Busard has shown that there is an almost literal agreement between Adelard I and the fragment of a Syriac redaction of book I which has come down to us. [Note 38: Busard (note 19 [i.e. [Busard 1983] — Z.K.), pp. 18–19.] Two theorems compared in detail by R. Lorch [Note 39: See note 22 [i.e. [Lorch 1987] — Z.K.] reveal, with one uncomfortable exception, affinities with the Hajjaj phraseology. But further investigations by Kunitzsch [Note*

40: Kunitzsch (note 21 [i.e. [Kunitzsch 1985] — Z.K.], p. 119.] give quite contrary indications, i.e., that Adelard I depends on Ishaq-Thabit rather than Hajjaj. Kunitzsch has noted that Adelard I has some sections of a complete literalness against the Arabic, while other sections show a high degree of ‘literary Latin’ transformation. Clearly, the text has been reworked.”

M. Folkerts lists the relevant manuscripts containing the Adelard I version stored in seven academic libraries in various European countries:

1. Bruges, Stadsbibliotheek 529 in Bruges, (f. 1–48v, from the XIII century), contains the books I–VIII.25,
2. Burney 275 from the British Library in London, (f. 302–308, from the XIV century), contains the text of VII.3–VIII.25,
3. D’Orville 70 in Bodleian Library, Oxford, (f. 39–71v, from the XIII/XIV centuries, X.36– XV.2,
4. Trinity College 47 in Oxford, (f. 171–180v, 163–170v, 155–162v, 147–154v, 139–146v, from the XII century, the oldest manuscript of the version), the books I– VIII.22,
5. Lat. 16201 from Bibliothèque Nationale in Paris, (f. 35–82, from XII century), the books I–VIII.22,
6. Regin. lat. 1137 in Vaticano, (f. 73v–74, from XIII century), contains the text of X.24 and X.17,
7. Österreichische Nationalbibliothek 83 in Wien, (f. 64–65v), from XIV century), the theorems X.36–49.

The next group of manuscripts represents the most influential and popular Latin version of the *Elements* in the Middle Ages starting from the end of the XII century. This version was displaced by the Campanus of Novara’s version in the XIII century, which is itself based on it. The number of relevant manuscripts indicates the predominant character of the Adelard II version. Folkerts lists 59 manuscripts, however, in [Busard 1996] are listed 61 mss. The oldest manuscripts with the proofs are Chartres 498 and the aforementioned Ms Trinity College 47 in Oxford. The critical edition of the text is published in [Busard 1983], [Busard 1996]. According to Busard, the most probably compiler of the version was Robert of Chester. It is important to note that Robert of Chester has his own unique formulations of the following theorems from the book I: I.1–3, 9, 12, 35–37 and 41. Busard and Folkerts indicate that the direction for proofs was incorporated into the previous text later, around 1160 A.D. together with the first diagrams in this version. In general, the text contains no proofs but only some indications how to perform them. Folkerts writes about this version as follows:

*“There are some positive reasons to show that the proofs in Adelard II were probably not original to Euclid. When we look at the earliest manuscripts of the text, we find that several of them contain no proofs at all, or at most very*

general, short indications how one might prove the proposition. In fact, there are about ten manuscripts written before the end of the twelfth century, but only two of them contain proofs beyond book VI. When proofs are given they are of various lengths and natures. Further, in some manuscripts one finds the proofs before the enunciations, in others they are beside the enunciation; in yet others there is a proof in the regular position but also an indication in the manner just spoken of in the margin.

At this point it is perhaps sensible to mention two particular manuscripts: München CLM 13021, and Paris BN Lat. 10257. CLM 13021 was written in Prüfening near Regensburg, probably in the sixties of the twelfth century, and contains one of the earliest witnesses to the Adelard II text. The first part of the text in this manuscript is similar to the Greek-Latin tradition associated with Boethius; but from book IV on it is Adelard II. There are no proofs. There is another copy of the same text in ms München CLM 23511, from the end of the twelfth century. The next manuscript mentioned above (Paris BN Lat. 10257) was originally in Chartres and also comes from the twelfth century; it is especially important in our story. [Note 44: See G. D. Goldat, *The Early Medieval Traditions of Euclid's The Elements* (unpublished dissertation: University of Wisconsin, 1957).] This manuscript has not only the Boethian excerpts in a contaminated form but also enunciations for all fifteen books. A notable feature of this work is the presence of what appears to be transcriptions of Greek terms for the various irrational quantities in book X. In book I–III the Paris manuscript shows some resemblance to the Munich manuscript and to an Oxford manuscript (Digby 98) which has an incomplete text. In the margin of some manuscripts containing Adelard II there are enunciations of ms Paris BN Lat. 10257 which are referred to by the words 'alia translatio'. In the Chartres manuscript now in Paris there are very short indications of proofs in the margin near the figures, and these are only for parts of book I. These excepted, there are eleven other proofs in book I which are somewhat longer. They differ from the corresponding passages in Adelard II, but they are arranged in the same way.

Another manuscript from Chartres is ms 497/498, the *Eptateuchon* written by Thierry of Chartres in about A.D. 1140, now unfortunately destroyed. It seems to have contained the oldest witness to Adelard II. It comprised the most modern texts of its time for the seven liberal arts, including both a text in the Greek-Latin Boethius tradition, the so-called *Geometry II* and also originally at least a part of the text of Adelard II. Some folios had been lost, but book VII–IX and the fragment of book XIV–XV contain no proofs, and the text fully confirms what we have deduced above from other manuscripts. The contents of this codex leads us to suppose that perhaps someone or some group from this part of France, maybe even Chartres itself, could have been

*responsible for making the original collection that later formed what we know as 'Adelard II'. (...)*

*The proofs in the various manuscripts of Adelard II are sometimes very different. Usually the content is the same, but the formulation is sometimes radically different. The proofs often contain interesting Arabic transcriptions and other traces of Arabic origin. There are more Arabic words in the proofs than in the text of the enunciations, but I think these words too can be explained without the assumption that there was a translation from the Arabic other than that by Hermann or Adelard I. The fact that in many cases the proofs are not in agreement with each other and that there are abridged or additional proofs in some manuscripts seems to indicate that the proofs were not written by only one person. I think that originally indications of proofs were given in the margin similar to those in ms Paris BN Lat. 10257 and in the Adelard I manuscripts Bruges 529 and Oxford Trinity College 47."*

The best manuscripts containing the directions to the proofs of the version are:

1. Dd. XII.61 (= 778) from the Cambridge University Library, (f. 2v–124v, from the XII (ex.) century), containing the books I – XV,
2. Universitni Knihovna III.H.19 (= 572) in Prague, (f. 1–127v, from the XII century, the books I–XII.4.

There are eleven manuscripts (including variants **Og<sub>1</sub>** and **Og<sub>2</sub>** below) with the next, Adelard III, version (compiled around 1200 A.D.); cf. [Folkerts 1989] and [Busard 2001]:

1. Medical Library 24 in Boston, **Bo** in [Busard 2001], the pages 110v–111r, (dated after 1466 A.D.), contains only the introduction,
2. Burney 275 from the British Library in London, **Lb** in [Busard 2001], (f. 308r–335r, from the XIV century), contains the books IX–XV.3,
3. Royal 15 B IV from the British Library in London, **Lf** in [Busard 2001], (f. 158r–167v, from the XIII century), contains X.94–XI.37,
4. Balliol College 257 in Oxford, **Ba** in [Busard 2001], (f. 3r–98v), from the XII (ex) century), contains the books I.1–XV.3,
5. Digby 174 from the Bodleian Library in Oxford, **Og** in [Busard 2001], (f. 99r–132v, from the XII century), contains the text of: I–XI.1), (f. 139r–145r with the text of Books X, Definitions – XI,1, **Og<sub>2</sub>** in [Busard 2001]), (f. 146r–153r with the text of Books II, Definitions – V, Definition 6, **Og<sub>1</sub>** in [Busard 2001]), VI.11–X (f. 154–159v),
6. D'Orville 70 from the Bodleian Library in Oxford, **Oc** in [Busard 2001], (f. 23v–38r, from XIII century), contains the books VII.7–X.36,
7. Savile 19 from the Bodleian Library in Oxford, (f. 1–37, from XIII century), contains the books I–VI (definitions only) of the so-called Adelard IIIB

version, the other nine manuscripts belongs to the Adelard IIIA version, (This manuscript is absent in [Busard 2001].)

8. Lat. 16648 from the Bibliothèque Nationale in Paris, **Pf** in [Busard 2001], (f. 2r–58r, from XIII century), contains the books X–XV,

9. Fondo antico 332 (= 1647) in the Biblioteca Nazionale Marciana in Venezia, **Vf** in [Busard 2001], (f. 86r–233r, from the XIII century), contains the books I–XV mixed with Adelard II version,

10. Österreichische Nationalbibliothek 83 in Wien, **Vi** in [Busard 2001], (f. 53r–63v, from XIV century), contains the text of the books VII.7–X.36.

Folkerts writes about the Adelard III version:

*“The version called Adelard III by Clagett has the same enunciations as Adelard II, which it cites but different and fuller proofs. Although it contains Arabic terms not in Adelard II, it is in all probability a commentary rather than an independent translation. It was written or compiled probably at the end of the twelfth century; the only manuscript which transmits the complete text, Oxford, Balliol College 257, seems to have been written in the first half of the thirteenth century.”*

There is a critical edition of the Adelard III version; cf. [Busard 2001]. This version cites the Adelard II version<sup>4</sup> to which it refers with the phrase “in commento.” Roger Bacon mentions the Adelard III version and he explicitly ascribes it to Adelard of Bath; cf. Roger’s Bacon *Communium Mathematica*.<sup>5</sup>

Some parts of the both versions, Adelard II and III, belong to the Boethian tradition; see above. J. E. Murdoch writes: “Adelard III also shows a direct awareness of this non-Euclidean arithmetical tradition by frequently citing Boethian terms for their Euclidean equivalents (e.g. in VII, def. 16-17 [=def. 18-19 Adel.])”<sup>6</sup>

The next group of manuscripts which stems from the Adelard II tradition is the reworking of Campanus of Novara. A *terminus ad quem* is derived from the earliest manuscript of this version, i.e. the codex from Florence, the Biblioteca Nazionale Centrale Magliabecch. XI 112, (f. 1-160, books I–XV), dated 1259 A.D. There is a critical edition of the text by Busard, cf. [Busard 2005].

*“He took over the enunciations in Adelard II, but the proofs do not correspond. (...) In this material is to be found Nayrizi’s commentary of the The Elements and, above all, Jordanus’ ‘Arithmetica’—the latter is particularly notable in the definitions of books VII and VIII. (...) It is well-known that in*

<sup>4</sup> See for instance the proof of the theorem VI. 27 (VI. 28 in Campanus’ version). Murdoch writes: “Adelard in Version III follows, and was aware of, more than one version of Adelard II”; cf. [Murdoch 1968], p. 72, note 14.

<sup>5</sup> Cf. [Murdoch 1968], p. 71, note 11 and [Clagett 1953] p. 23, note 18. Cf. also [Busard 1985], p. 135.

<sup>6</sup> See [Murdoch 1968], p. 73, note 15.

1482 the Campanus version was printed in Venice—this was, except of editions of fragments of the Boethian tradition, the first printed Euclid in the West— that in 1498 Giorgio Valla published books XIV and XV together with commentaries and took over Euclidean material into his encyclopedic ‘*De expetendis et fugiendis rebus*’ (Venice, 1501), and that Bartolomeo Zamberti published a new Latin translation from the Greek in 1505. But we know from two Campanus manuscripts [Note 58: Stuttgart HB XI 24, and Vat. Palat. lat. 1352.] that already in 1450 Pope Nicholas V, who was also responsible for the new Archimedes translation made by Jacobus Cremonensis, assembled a Greek Euclid text. Some years later Regiomontanus tried to reconstruct the original Euclid text with the help of Bessarion’s Greek manuscripts. At least the first books of Euclid’s *The Elements* were taught within the university curriculum of the *artes liberales*. There are some manuscripts of the fifteenth century which seem to be notes or copies of students, but up to now there is no systematical research on them.

(...) [It was] [f]irst edited by E. Ratdolt (Venice, 1482). Later editions by Henricus Stephanus (Paris, 1516); J. Hervagius (Basel, 1537, 1546, 1558) and others” cf. [Folkerts1989].

Thus, Campanus took the Adelard II enunciations, made some new or completed old proofs. Campanus completed the text of the *Elements* with the theorems X.13, X.16, X.24, XII.6, XII.13 (in Heiberg’s numbering)<sup>7</sup>. He also added some new axioms taking them mainly from Jordanus’ *Arithmetica*. It is necessary to note that Campanus’ edition became the standard and most popular text of the *Elements* from 1260 to the sixteenth century.

The next group in the Adelard II tradition is formed by seven manuscripts of a commentary on the *Elements*. These manuscripts contain parts of Adelard II text mixed with the (Pseudo-)Boethian *Geometrie II*. They can be divided into two other sublines; to the first one only one manuscript belongs: Lüneburg, Ratsbibliothek Misc. D 4o 48, (f. 13–17v, dated ca. 1200 A.D., contains books I–IV). The author of this manuscript used the *Geometrie II* for the definitions and postulates and he used the Adelard II in the case of the axioms and enunciations; cf. [Folkerts 1970a] and [LoBello 2003a] p. 59. The other (more important) manuscripts are listed by Folkerts in [Folkerts 1989]:

1. Bibliothek der Rijksuniversiteit in Leiden, Voss. lat. qu. 92 (f. 2–2v, 1–1v, from XIIth century (ex.), contains the fragments of the book III. 34–36 (i.e. in Heiberg’s numbering III. 33–35), theorems IV. 14–16 (Heiberg’s IV. 14–19), both form the Adelard II version, mixed with additions from (Pseudo-)Boethius *Geometry II*,

<sup>7</sup> These theorems are missing in the Adelard II and III as well as in the Hermann’s, Robert’s, Gerard’s versions. Therefore he probably used a manuscript directly translated from the Greek, e.g. that edited by Murdoch in [Murdoch 1967].

2. Bayerische Staatsbibliothek in München, CLM 13021, (f. 164–169v, from XII century, contains books I–III),
3. Bodleian Library in Oxford, Digby 98, (f. 78–85v, from the XII century, contains books I – III def.4),
4. Bibliothèque Nationale in Paris, Lat. 10257, (f. 1–88, from the XII century, contains books I–XV),
5. (olim) San Juan Capistrano, Honeyman MS 50 (MS Math.1), (f. 1–25, from the XII century, contains books I–XV which are copied from Paris BN Lat. 10257).

The manuscript Lat. 10257 from Paris contains the text of the *Geometrie II* (without the three “standard” Boethian proofs) and then he added indications to and outlines of (many) proofs which are more complete than these of Adelard II.

The last group related to the Adelard II version is the so-called “V” line of manuscripts; cf. a critical edition of the text in [Busard 1996a]. There are five manuscripts of it:

1. Universitätsbibliothek in Bonn, S 73, (f. 1–86, from the XIII century, contains books I–XV),
2. Bibliothèque Nationale in Paris, Lat. 7292, (f. 188–245v, from the XV, contains books I–VI),
3. Biblioteca Fabroniana in Pistoia, 315, (f. 1–48v, from the XIV century, contains books I–XV),
4. Vaticano, Regin. lat. 1268, (f. 1–69, from the XIV century (in.), contains books I–XV; the text contains also the Boethian enunciations (interpolated by the other hand) of the theorems I.2, I.3, I.6, I.7, I.9 and I.10),
5. Österreichische Nationalbibliothek in Wien, 5304, (f. 1–126v, from the XVI century, contains books I. 1–XV).

The composer of the V version (written around 1250 A.D.) used the compilation made by Robert of Chester (i.e. Adelard II version) for the formulation of definitions, postulates, axioms, and enunciations. However, he performed his own explicit proofs of the theorems in which he applied his own specific wording.

The last commentary of the *Elements* reviewed in this section, the commentary of Albertus Magnus (prepared around 1260 A.D.), is based on the above commentary in some parts with the definitions, postulates and the enunciations of the theorems copied verbatim from the V version. Moreover, this commentary is independent from the Campanus’ edition. We have one manuscript of the commentary of Albertus Magnus: Dominikanerkloster 80/45, Wien (f. 105–145, from the XIII century,<sup>8</sup> contains the commentary to the books I–IV). There is a

<sup>8</sup> M. Folkerts in [Folkerts 1989] dated the manuscript on the XIV century, however, some paleographic properties of the text indicates on the XIII century; see LoBello 2003c].

critical edition of the book I by P. M. J. E. Tummers; cf. [Tummers 1984], see also an English translation in [LoBello 2003c].

Albertus Magnus was not an especially original commentator. He simply used the commentary of al-Nayrizi translated by Gerard of Cremona. When he tried to establish his own proof, he was usually mistaken. His original contribution is limited to some occasional remarks and to the composition of the preface.

\* \* \*

We should note that there is a change in the intuitive foundations of mathematics concerning the creation of mathematical entities from their principles. One can see how *motion* (translations, superpositions, incursions of sides of figures, etc.) enters gradually geometry and how this is a new element in geometry in comparison to Plato's "static" way of thinking. Simplicius, al-Nayrizi, Albertus Magnus speak about the movements of a point, a line, a circle, a surface, a body. A line is a principle of a surface *because* when it is moved in the second dimension, it produces a surface, etc. The motion is predominant in Albertus Magnus' *Commentary on the first book of the Euclid's Elements of geometry*. In the last commentary also the concept of *space* is used, and Albert even speaks about a point as if it was a part of space:

*Motion, however, is not continuous except from the space over which it occurs, and time gets its continuity from motion, and the being of motion and of time is continuous from space, and a bit of motion and an instant of time are indivisible from the indivisible element of space, which is the point;* (see [LoBello 2003c], pp. 4–5).

From one Simplicius' fragments preserved in the Arabic and Latin texts of al-Nayrizi's commentary, it is clear that Simplicius locates the realm of mathematical objects in the realm of imagination. Let us remind the reader that for Aristotle every geometrical object has to be represented by a real property of a real, physical object (substance, body). Simplicius and his followers: the Arabs, Gerard, Albertus Magnus, can see that postulates may be not realizable in the real world. Therefore, Simplicius and the Arabs introduce the concept of an imagined mathematical object. The conflict with reality is especially sharp with respect to infinite objects even if they are thought as only potentially infinite. The realm of mathematics *exceeds* the reality. For Simplicius and medieval authors, this "exceed" is apparent only in some secondary points, mainly concerning the possibility of unbounded extension of some objects, mainly some lines and surfaces. However, this transgression of reality is seen as the main obstacle in understanding of geometry by students and people uneducated in geometry. The above partial and seemingly harmless (because only imagined), *separation* of geometrical objects from the reality allowed al-Nayrizi to operate

with the *two* different concept of lines: finite and *actually infinite*. Though there is no infinite line in reality, it can nevertheless exist in pure imagination. The above ideas are crucial for the later development of geometry.

Moreover, Albertus Magnus operates in geometry with the concept of *imaginary* infinite space (which is absent from ancient Euclidean geometry) and such an object does not exist in the real world. The concept of an infinite space is absent in other Arabic and Latin sources. Albert, in an unintended way, *changes* the original meaning of one al-Nayrizi's remark.

Al-Nayrizi postulates the existence of *geometrical matter*:

*As for this postulate [i.e. the first – Z.K.], it is necessary to ask that it be postulated because the existence of geometrical matter is in the imagination. For, indeed, if their existences were in material bodies, it would be rash to postulate that a straight line be drawn from Aries to Libra. (Cf. Codex Leidensis, [LoBello 2003a], p. 92.)*

*And this by necessity had to be posited, because the existence of geometrical matter consists in the imagination [quod essentia materie geometrie consistit in imaginatione]. For if it were in bodies having matter, it would be superfluous that it be asked to be postulated that a straight line be drawn from Aries to Libra. (Cf. Gerard, [LoBello 2003b], p. 45; Heiberg-Curtze 31.1–5.)*

Albert is more explicit in saying that there is no “geometrical matter” because it is an unacceptable thing in his nominalistic Aristotelian philosophy:

*Deceived, therefore, are they who said that they [i.e. the postulates – Z.K.] are postulated for no other reason than that geometrical matter be generated through them, namely, because all geometry revolves around imaginable quantity and not the sensible continuum. (Cf. [LoBello 2003c], pp. 23–24.)*

Al-Nayrizi indicates also some new axioms which were introduced to geometry after Euclid. Pappus is counted among the developers of new axioms. One of his axiom is:

*We shall need this result in the first figure [i.e. in the first theorem – Z.K.]: With regard to the straight line and the plane surface, it is possible, because of their levelness, for them to be extended with an infinite extension, forever. (Cf. [LoBello 2003a], p. 104; cf. also Proclus' *Commentary on the first book of Euclid's Elements*, 198.6–10. The above comment is explicitly attributed to Pappus by al-Nayrizi.)*

*[A]nd it is possible for a plane surface and a straight line, for this reason, because they are plane, to be extended in infinitum [in infinitum protrahi]. (Cf. Gerard, [LoBello 2003b], p. 54, Heiberg-Curtze 38.10–15; Pappus is not explicitly indicated.)*

Albert adds that “*infinitely many others* [i.e. common notions – Z.K.] *can be added*” (cf. [LoBello 2003c], p. 30) and he lists some of them. He writes, (*ibidem*):

*For magnitude decreases in infinitum. Among numbers, however, if the prior should be a submultitude of the second, whatever third will be equally a submultitude of some fourth. Multitude increases in infinitum.*

The way of the *Elements* (cultural) transmission from the ancient time to the modernity influenced the nowadays adopted their content. As it was indicated above, the historical description of the sources indicates the aprioricity of mathematics because the lost sections of the text of the *Elements* were reconstructed, the text and some lacunas were supplemented in an essential way with new axioms, lemmas, theorems and comments. In many cases, the content of such additions was *the same* as, or equivalent to, the original ancient text.

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ABOUT THE AUTHOR — professor of the Warsaw University of Technology (Department of the Philosophy of Science, Sociology and the Foundations of Technology, Faculty of Administration and Social Science), and of the Institute of Philosophy and Sociology of the Polish Academy of Sciences (Department of the Philosophy and Hermeneutics of Mathematics, Department of Theory of Cognition and Philosophy of Science, and Department of the Researches of Ancient Philosophy and the History of Ontology). His research concerns the philosophy of mathematics and natural sciences, history of mathematics and natural sciences, logic, mathematics, ontology and epistemology.

